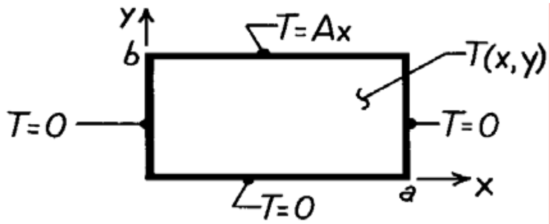
## PROBLEM 4.4

**KNOWN:** Rectangular plate subjected to prescribed boundary conditions.

**FIND:** Steady-state temperature distribution.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) Steady-state, 2-D conduction, (2) Constant properties.

**ANALYSIS:** The solution follows the method of Section 4.2. The product solution is

$$T(x,y) = X(x)\cdot Y(y) = (C_1\cos\lambda x + C_2\sin\lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right)$$

and the boundary conditions are: T(0,y) = 0, T(a,y) = 0, T(x,0) = 0, T(x.b) = Ax. Applying BC#1, T(0,y) = 0, find $C_1 = 0$. Applying BC#2, T(a,y) = 0, find that $\lambda = n\pi/a$ with n = 1,2,…. Applying BC#3, T(x,0) = 0, find that $C_3 = -C_4$. Hence, the product solution is

$$T(x,y) = X(x)\cdot Y(y) = C_2 C_4 \sin\left[\frac{n\pi}{a}x\right]\left(e^{+\lambda y} - e^{-\lambda y}\right).$$

Combining constants and using superposition, find

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi x}{a}\right]\sinh\left[\frac{n\pi y}{a}\right].$$

To evaluate $C_n$ and satisfy BC#4, use orthogonal functions with Equation 4.16 to find

$$C_n = \int_0^a Ax\cdot\sin\left[\frac{n\pi x}{a}\right]\cdot dx/\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\left[\frac{n\pi x}{a}\right]dx,$$

noting that y = b. The numerator, denominator and $C_n$, respectively, are:

$$A\int_0^a x\cdot\sin\frac{n\pi x}{a}\cdot dx = A\left[\left[\frac{a}{n\pi}\right]^2\sin\left[\frac{n\pi x}{a}\right] - \frac{ax}{n\pi}\cos\left[\frac{n\pi x}{a}\right]\right]_0^a = \frac{Aa^2}{n\pi}\left[-\cos(n\pi)\right] = \frac{Aa^2}{n\pi}(-1)^{n+1},$$

$$\sinh\left[\frac{n\pi b}{a}\right]\int_0^a \sin^2\frac{n\pi x}{a}\cdot dx = \sinh\left[\frac{n\pi b}{a}\right]\left[\frac{1}{2}x - \frac{a}{4n\pi}\sin\left[\frac{2n\pi x}{a}\right]\right]_0^a = \frac{a}{2}\cdot\sinh\left[\frac{n\pi b}{a}\right],$$

$$C_n = \frac{Aa^2}{n\pi}(-1)^{n+1}/\frac{a}{2}\sinh\left[\frac{n\pi b}{a}\right] = 2Aa\,(-1)^{n+1}/n\pi\,\sinh\left[\frac{n\pi b}{a}\right].$$

Hence, the temperature distribution is

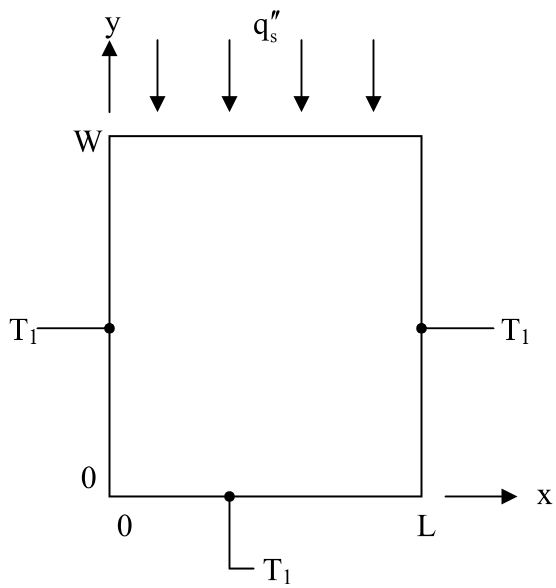
$$T(x,y) = \frac{2\,Aa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin\left[\frac{n\pi x}{a}\right]\frac{\sinh\left[\frac{n\pi y}{a}\right]}{\sinh\left[\frac{n\pi b}{a}\right]}.$$ <

# PROBLEM 4.5

**KNOWN:** Boundary conditions on four sides of a rectangular plate.

**FIND:** Temperature distribution.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) Two-dimensional, steady-state conduction, (2) Constant properties.

**ANALYSIS:** This problem differs from the one solved in Section 4.2 only in the boundary condition at the top surface. Defining $\theta = T - T_\infty$, the differential equation and boundary conditions are

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\theta(0,y) = 0 \quad \theta(L,y) = 0 \quad \theta(x,0) = 0 \quad k\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = q''_s \qquad (1a,b,c,d)$$

The solution is identical to that in Section 4.2 through Equation (4.11),

$$\theta = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L} \qquad (2)$$

To determine $C_n$, we now apply the top surface boundary condition, Equation (1d). Differentiating Equation (2) yields

Continued….

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=W} = \sum_{n=1}^{\infty} C_n \frac{n\pi}{L} \sin\frac{n\pi x}{L} \cosh\frac{n\pi W}{L} \tag{3}$$

Substituting this into Equation (1d) results in

$$\frac{q_s''}{k} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{4}$$

where $A_n = C_n(n\pi/L)\cosh(n\pi W/L)$. The principles expressed in Equations (4.13) through (4.16) still apply, but now with reference to Equation (4) and Equation (4.14), we should choose $f(x) = q_s''/k$, $g_n(x) = \sin\frac{n\pi x}{L}$. Equation (4.16) then becomes

$$A_n = \frac{\frac{q_s''}{k}\int_0^L \sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} = \frac{q_s''}{k}\frac{2}{\pi}\frac{(-1)^{n+1}+1}{n}$$

Thus

$$C_n = 2\frac{q_s''L}{k}\frac{(-1)^{n+1}+1}{n^2\pi^2\cosh(n\pi W/L)} \tag{5}$$

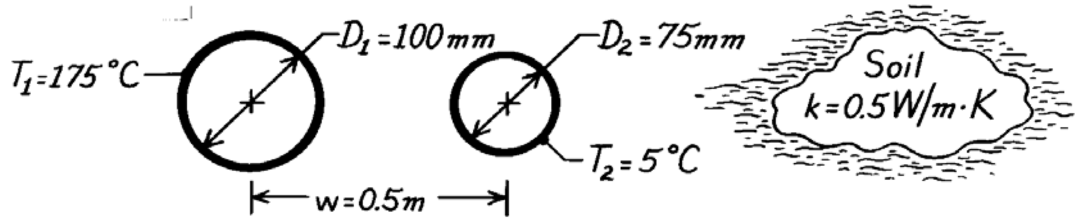
The solution is given by Equation (2) with $C_n$ defined by Equation (5).

**PROBLEM 4.13**

**KNOWN:** Surface temperatures of two parallel pipe lines buried in soil.

**FIND:** Heat transfer per unit length between the pipe lines.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) Pipe lines are buried very deeply, approximating burial in an infinite medium, (5) Pipe length >> $D_1$ or $D_2$ and w > $D_1$ or $D_2$.

**ANALYSIS:** The heat transfer rate per unit length from the hot pipe to the cool pipe is

$$q' = \frac{q}{L} = \frac{S}{L} k \left(T_1 - T_2\right).$$

The shape factor S for this configuration is given in Table 4.1 as

$$S = \frac{2\pi L}{\cosh^{-1}\left[\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right]}.$$

Substituting numerical values,

$$\frac{S}{L} = 2\pi / \cosh^{-1}\left[\frac{4\times(0.5\text{m})^2 - (0.1\text{m})^2 - (0.075\text{m})^2}{2\times 0.1\text{m}\times 0.075\text{m}}\right] = 2\pi / \cosh^{-1}(65.63)$$

$$\frac{S}{L} = 2\pi / 4.88 = 1.29.$$

Hence, the heat rate per unit length is

$$q' = 1.29 \times 0.5\text{W/m}\cdot\text{K}\left(175 - 5\right)^\circ\text{C} = 110\ \text{W/m}.$$ <

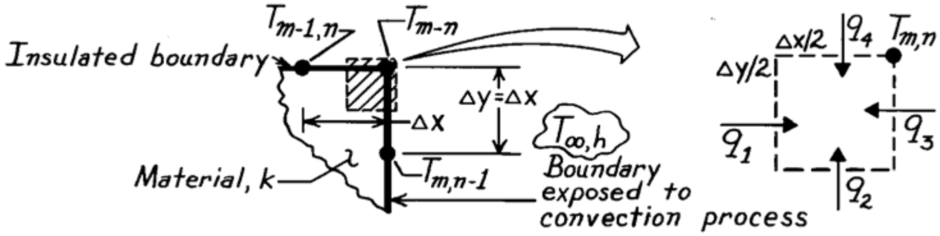
**COMMENTS:** The heat gain to the cooler pipe line will be larger than 110 W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

# PROBLEM 4.34

**KNOWN:** External corner of a two-dimensional system whose boundaries are subjected to prescribed conditions.

**FIND:** Finite-difference equations for these situations: (a) Upper boundary is perfectly insulated and side boundary is subjected to a convection process, (b) Both boundaries are perfectly insulated; compare result with Eq. 4.43.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) No internal generation.

**ANALYSIS:** Consider the nodal point configuration shown in the schematic and also as Case 4, Table 4.2. The control volume about the node – shaded area above of unit thickness normal to the page – has dimensions, $(\Delta x/2)(\Delta y/2)\cdot 1$. The heat transfer processes at the surface of the CV are identified as $q_1$, $q_2$ $\cdots$. Perform an energy balance wherein the processes are expressed using the appropriate rate equations.

(a) With the upper boundary insulated and the side boundary subjected to a convection process, the energy balance has the form

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_3 + q_4 = 0 \tag{1,2}$$

$$k\left[\frac{\Delta y}{2}\cdot 1\right]\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left[\frac{\Delta x}{2}\cdot 1\right]\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left[\frac{\Delta y}{2}\cdot 1\right]\left(T_{\infty} - T_{m,n}\right) + 0 = 0.$$

Letting $\Delta x = \Delta y$, and regrouping, find

$$T_{m,n-1} + T_{m-1,n} + \frac{h\Delta x}{k}T_{\infty} - 2\left[\frac{1}{2}\frac{h\Delta x}{k} + 1\right]T_{m,n} = 0. \tag{3}$$ <

(b) With both boundaries insulated, the energy balance of Eq. (2) would have $q_3 = q_4 = 0$. The same result would be obtained by letting $h = 0$ in the finite-difference equation, Eq. (3). The result is

$$T_{m,n-1} + T_{m-1,n} - 2T_{m,n} = 0.$$ <

Note that this expression is identical to Eq. 4.43 when $h = 0$, in which case both boundaries are insulated.

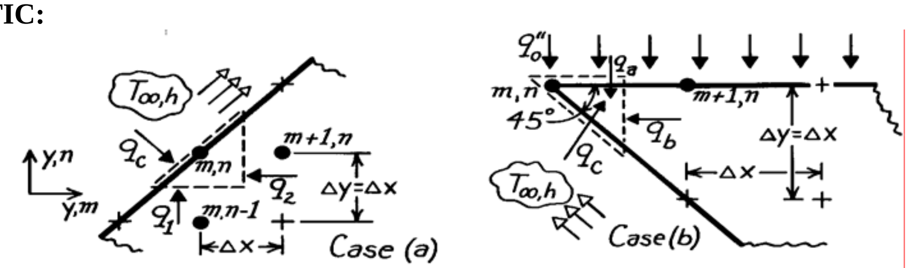
**COMMENTS:** Note the convenience resulting from formulating the energy balance by *assuming* that all the heat flow is *into the node.*

# PROBLEM 4.39

**KNOWN:** Nodal point configurations corresponding to a diagonal surface boundary subjected to a convection process and to the tip of a machine tool subjected to constant heat flux and convection cooling.

**FIND:** Finite-difference equations for the node m,n in the two situations shown.

**SCHEMATIC:**




**ASSUMPTIONS:** (1) Steady-state, 2-D conduction, (2) Constant properties.

**ANALYSIS:** (a) The control volume about node m,n has triangular shape with sides $\Delta x$ and $\Delta y$ while the diagonal (surface) length is $\sqrt{2}\ \Delta x$. The heat rates associated with the control volume are due to conduction, $q_1$ and $q_2$, and to convection, $q_c$. Performing an energy balance, find

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_1 + q_2 + q_c = 0$$

$$k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h\left(\sqrt{2}\ \Delta x \cdot 1\right)\left(T_\infty - T_{m,n}\right) = 0.$$

Note that we have considered the solid to have unit depth normal to the page. Recognizing that $\Delta x = \Delta y$, dividing each term by k and regrouping, find

$$T_{m,n-1} + T_{m+1,n} + \sqrt{2}\ \cdot \frac{h\Delta x}{k} T_\infty - \left[2 + \sqrt{2}\ \cdot \frac{h\Delta x}{k}\right] T_{m,n} = 0.$$ <

(b) The control volume about node m,n has triangular shape with sides $\Delta x/2$ and $\Delta y/2$ while the lower diagonal surface length is $\sqrt{2}(\Delta x/2)$. The heat rates associated with the control volume are due to the constant heat flux, $q_a$, to conduction, $q_b$, and to the convection process, $q_c$. Perform an energy balance,

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q_a + q_b + q_c = 0$$

$$q_o'' \cdot \left[\frac{\Delta x}{2} \cdot 1\right] + k \cdot \left[\frac{\Delta y}{2} \cdot 1\right] \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h \cdot \left[\sqrt{2} \cdot \frac{\Delta x}{2}\right]\left(T_\infty - T_{m,n}\right) = 0.$$

Recognizing that $\Delta x = \Delta y$, dividing each term by k/2 and regrouping, find

$$T_{m+1,n} + \sqrt{2} \cdot \frac{h\Delta x}{k} \cdot T_\infty + q_o'' \cdot \frac{\Delta x}{k} - \left(1 + \sqrt{2} \cdot \frac{h\Delta x}{k}\right) T_{m,n} = 0.$$ <

**COMMENTS:** Note the appearance of the term $h\Delta x/k$ in both results, which is a dimensionless parameter (the *Biot number*) characterizing the relative effects of convection and conduction.