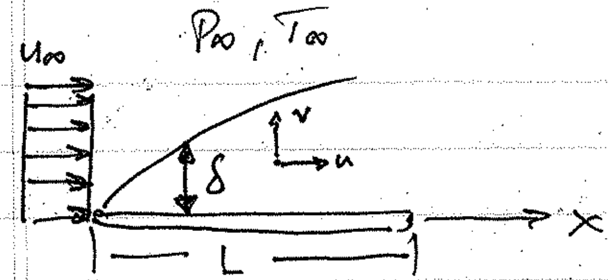
BOUNDARY LAYER EQS. - VELOCITY & THERMAL



$$\left.\begin{array}{l} u \sim u_\infty \\ x \sim L \\ y \sim \delta \\ P \sim P_\infty \\ T \sim T_\infty \end{array}\right\} \text{scales of } u, \delta x, \delta y$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{u_\infty}{L} + \frac{v}{\delta} = 0 \qquad \underline{\text{Eq. 1}}$$

$$\frac{u_\infty}{L} = -\frac{v}{\delta}$$

$$\frac{u_\infty}{L} \sim \frac{v}{\delta} \qquad \underline{\text{Eq. 2}}$$

X-Momentum Eq.:

$$\overset{①}{u\frac{\partial u}{\partial x}} + \overset{②}{v\frac{\partial u}{\partial y}} = \overset{③}{-\frac{1}{\rho}\frac{\partial P}{\partial x}} + \nu\left(\overset{④}{\frac{\partial^2 u}{\partial x^2}} + \overset{⑤}{\frac{\partial^2 u}{\partial y^2}}\right) \qquad \underline{\text{Eq. 3}}$$

$$① \; u\frac{\partial u}{\partial x} \sim u_\infty \frac{u_\infty}{L} = \frac{u_\infty^2}{L} \qquad ② \; v\frac{\partial u}{\partial y} \sim v\frac{u_\infty}{\delta} \sim \frac{u_\infty^2}{L}$$

both convective terms must stay (both of order $\frac{u_\infty^2}{L}$)

$$③ \; -\frac{1}{\rho}\frac{P_\infty}{L} \qquad ④ \; \nu\frac{u_\infty}{L^2} \qquad ⑤ \; \nu\frac{u_\infty}{\delta^2}$$

Assume B.L. is slender: $\delta << L$, $\cancel{\nu\frac{u_\infty}{(\delta/L)^2}\frac{1}{L^2} >>}$ "Get rid of ④"

$$\overset{⑤}{\nu\frac{u_\infty}{L^2}\frac{1}{(\delta/L)^2}} >> \overset{④}{\nu\frac{u_\infty}{L^2}}$$

Rewrite X-Momentum

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}}$$ Eq. 4

Y-Momentum

$$\overset{①}{u\frac{\partial v}{\partial x}} + \overset{②}{v\frac{\partial v}{\partial y}} = \overset{③}{-\frac{1}{\rho}\frac{\partial p}{\partial y}} + \nu\left(\overset{④}{\frac{\partial^2 v}{\partial x^2}} + \overset{⑤}{\frac{\partial^2 v}{\partial y^2}}\right)$$ Eq. 5

$$① = u_\infty \frac{v}{L} = \frac{U_\infty}{L} u_\infty \frac{\delta}{L} = \frac{u_\infty^2}{L}\cdot\frac{\delta}{L}$$

① & ② of same order

$$② = \frac{v}{\delta} v = \frac{v^2}{\delta} = \frac{\delta^2 u_\infty^2}{L^2}\frac{1}{\delta} = \frac{u_\infty^2}{L}\frac{\delta}{L}$$

$$③ = -\frac{1}{\rho}\frac{P_\infty}{\delta}$$

$$④ = \nu\frac{\partial^2 v}{\partial x^2} = \nu\frac{U_\infty \delta}{L}\frac{1}{L^2} = \nu\frac{U_\infty}{L^2}\cdot\underbrace{\frac{\delta}{L}}_{\ll 1}$$

⑤ stays
④ goes away!

$$⑤ = \nu\frac{\partial^2 v}{\partial y^2} = \nu\frac{U_\infty \cancel{\delta}}{L}\cdot\frac{1}{\delta^{\cancel{2}}} = \nu\frac{U_\infty}{L^2}\underbrace{\frac{1}{\delta/L}}_{\gg 1}$$

Rewrite Y-Momentum

$$\boxed{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2}}$$ Eq. 6

Note, may write

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx}$$

Look at Eq (4) scale analysis

$$\underbrace{\frac{u_\infty^2}{L} + \frac{u_\infty^2}{L}}_{\frac{u_\infty^2}{L}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{u_\infty}{L^2}\frac{1}{(\delta/L)^2} \qquad \text{Eq. 7}$$

Look at Eq (6) scale analysis

$$\underbrace{\frac{u_\infty^2}{L}\frac{\delta}{L} + \frac{u_\infty^2}{L}\frac{\delta}{L}}_{\frac{u_\infty^2}{L}\frac{\delta}{L}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{u_\infty}{L^2}\frac{1}{\delta/L} \qquad \text{Eq. 8}$$

Make same kind of comparison for 7 & 8

$$\frac{\partial p}{\partial x} \sim \mu \frac{u_\infty}{L^2}\frac{1}{(\delta/L)^2} = \frac{\mu u_\infty}{\delta^2}$$

$$\frac{\partial p}{\partial y} \sim \mu \frac{u_\infty}{L^2}\frac{1}{\delta/L} = \mu \overbrace{\frac{u_\infty}{L}}^{v/\delta}\frac{1}{\delta} = \frac{\mu v}{\delta^2}$$

$$\frac{(\partial P/\partial y)\left(\frac{dy}{dx}\right)}{\partial P/\partial x} = \frac{\cancel{\frac{\mu v}{\delta^2}} \cdot \overbrace{\frac{dy}{dx}}^{\delta/L}}{\cancel{\frac{\mu u_\infty}{\delta^2}}} = \frac{v\delta}{u_\infty L} = \left(\frac{\delta}{L}\right)^2 \ll 1$$

So... $\frac{dP}{dx} \sim \frac{\partial P}{\partial x}$ no need to worry about $\frac{\partial P}{\partial y}$ ...

X-Momentum: $\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}}$ The B.L. Equation (Eq. 9)

ENERGY EQ.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$u_\infty \frac{T_\infty}{L} + \frac{u_\infty \cancel{\delta}}{L}\frac{T_\infty}{\cancel{\delta}} = \alpha\left(\frac{T_\infty}{L^2} + \frac{T_\infty}{\delta^2}\right)$$

$$\frac{T_\infty}{L^2} + \frac{T_\infty}{L^2}\frac{1}{(\delta/L)^2}$$

big!

Energy goes to...

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

[$\frac{\partial^2 T}{\partial x^2}$ is small – physically this means rate of conduction in x-dir. is small]

Summarize B.L. Problem

(1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(2) $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

(3) $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

11/18/08

Comparison of B.L. thicknesses

$$\tau = \mu \frac{\partial u}{\partial y} \Rightarrow \tau \sim \mu \frac{u_\infty}{\delta}$$

what we want/need

need this to know about τ

Usually, $P_\infty \sim$ const., very small change in P_∞ vs x

$$\frac{dP_\infty}{dx} = 0$$

x-Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{u_\infty^2}{L} \qquad v\frac{u_\infty}{\delta} \qquad \nu\frac{u_\infty}{\delta^2}$$

$$\frac{v}{\delta} \sim \frac{u_\infty}{L}$$

$$v \sim \frac{u_\infty \delta}{L}$$

$$\frac{u_\infty^2}{L}$$

$$\frac{u_\infty^{\cancel{2}}}{L} \sim \nu \frac{\cancel{u_\infty}}{\delta^2}$$

$$\delta \sim \left[\frac{\nu L}{u_\infty}\right]^{1/2} \cdot \frac{L^{1/2}}{L^{1/2}}$$

$$\frac{\delta}{L} \sim \left(\frac{u_\infty L}{\nu}\right)^{-1/2}$$

$$\rho\frac{u_\infty L}{\mu} = Re_L$$

$$\boxed{\frac{\delta}{L} \sim Re_L^{-1/2}}$$

Assumed $\frac{\delta}{L} \ll 1$, so $Re_L^{1/2} \gg 1$ must be the case!

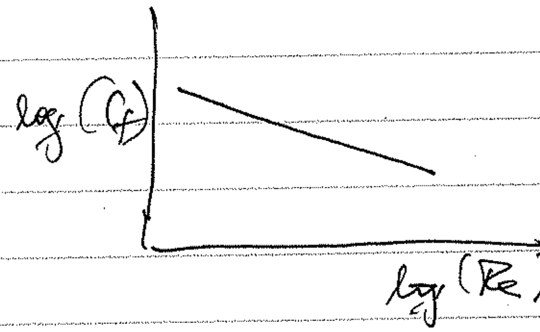
$$\tau \sim \mu \frac{u_\infty}{\delta} = \mu \frac{u_\infty}{L} Re_L^{1/2} \qquad \left(\rho \frac{u_\infty L}{\mu}\right)^{1/2}$$

$$= \rho u_\infty^2 Re_L^{-1/2}$$ Hmm...

$$C_f = \frac{\tau}{\frac{1}{2}\rho u_\infty^2}$$

$$C_f \sim \frac{\rho u_\infty^2 Re_L^{-1/2}}{\frac{1}{2}\rho u_\infty^2}$$

$$\boxed{C_f \sim Re_L^{-1/2}}$$



scale analysis result
(exact result will only give a fraction out front, i.e. $C_f = C_1 Re_L^{-1/2}$ (0–10))

$$h(T_s - T_\infty) = q'' = -k_f \frac{\partial T}{\partial y}\Big|_0$$

$$h = \frac{-k_f \, \partial T/\partial y|_0}{T_s - T_\infty} = -k_f \frac{\Delta T/\delta_T}{\Delta T}$$

$$h \sim \frac{k_f}{\delta_T}$$

From Energy Eq:

$$u_\infty \frac{T_\infty}{L} + v \frac{T_\infty}{\delta_T} = \alpha \frac{T_\infty}{\delta_T^2}$$

Mass cont. to δ_T B.L. $\quad \frac{u_\infty}{L} + \frac{v}{\delta_T} = 0$

$$v \sim \frac{u_\infty \delta_T}{L}$$

$$\cancel{u_\infty \frac{T_\infty}{L} \sim \alpha \frac{T_\infty}{\delta_T^2}}$$

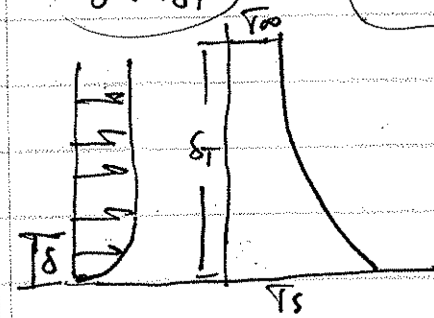
Energy Eq. to get δ_T:

$$u \frac{T_\infty}{L} + v \frac{T_\infty}{\delta_T} = \alpha \frac{T_\infty}{\delta_T^2}$$

From Continuity: $\frac{u_\infty}{L} + \frac{v}{\delta_T} = 0$

Note: $u \neq u_\infty$ necessarily

$\delta << \delta_T$



$u \sim u_\infty$ in this case

$$\frac{u_\infty}{L} + \frac{v}{\delta_T} = 0$$

$$\frac{u_\infty}{L} \sim v/\delta_T$$

ENERGY GIVES:

$$\frac{u_\infty \cancel{T_\infty}}{L} + \frac{u_\infty \cancel{\delta_T}\, \cancel{T_\infty}}{L\, \cancel{\delta_T}} = \alpha \frac{\cancel{T_\infty}}{\delta_T^2}$$

$$\delta_T \sim \left[\frac{\alpha L}{u_\infty}\right]^{1/2}$$

$$\cancel{\delta_T \sim \left(\frac{kL}{\rho c_p u_\infty}\right)^{1/2}}$$

$$\cancel{\sim \left(\frac{kL^2}{\mu \rho c_p}\right)^{1/2} \left(\frac{\mu}{\rho u_\infty L}\right)^{1/2}}$$

$$\delta_T \sim \underbrace{\left[\frac{\alpha}{\nu}\right]^{1/2}}_{Pr^{-1/2}} \underbrace{\left[\frac{\nu}{u_\infty L}\right]^{1/2}}_{Re_L^{-1/2}} \cdot L$$

$$Pr = \frac{\nu}{\alpha}$$ friction – viscous / thermal conduction

When $\delta << \delta_T$

$$\frac{\delta_T}{L} \sim Re_L^{-1/2} \, Pr^{-1/2}$$

fluid property

$Pr(H_2O @ 20°C) = 7$

$Pr(AIR @ 20°C) = 0.72$

$Pr(\underline{Liq. Na}) = 0.011$

(liquid metals)

Note: Recall $\frac{\delta}{L} \sim Re_L^{-1/2}$

$\frac{\delta_T}{\delta} \sim Pr^{-1/2}$ And assumed that $\delta << \delta_T$

so...

$$\underbrace{Pr^{-1/2} >> 1 \Rightarrow \frac{\delta_T}{\delta} >> 1}$$

$Pr^{1/2} << 1$

Really only true for <u>liquid metals</u>

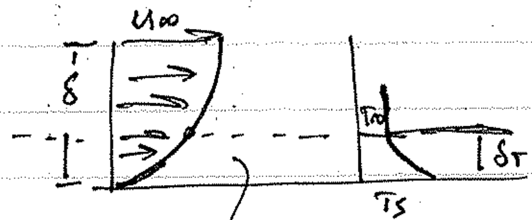
<u>In this case:</u>

$$h \sim \frac{k_f}{\delta_T} \sim \frac{k_f}{L\, Re_L^{-1/2} Pr^{-1/2}} \sim \frac{k_f}{L} Re_L^{+1/2} Pr^{+1/2}$$

$$Nu \equiv \frac{hL}{k}$$

$$\boxed{Nu \sim Re_L^{1/2} \, Pr^{1/2}}$$

Note: When $\delta >> \delta_T$



Only concerned with this region
Where $u \sim u_\infty \frac{\delta_T}{\delta}$

$$\frac{u_\infty \frac{\delta_T}{\delta}}{L} + \frac{v}{\delta_T} \Rightarrow$$
$$v \sim \frac{u_\infty \delta_T^2}{\delta L}$$

So...

$$u\frac{\cancel{T_\infty}}{L} + v\frac{\cancel{T_\infty}}{\delta_T} \sim \alpha \frac{\cancel{T_\infty}}{\delta_T^2}$$

$$u_\infty \frac{\delta_T}{\delta} \qquad \frac{u_\infty \delta_T^2}{\delta L}$$

$$u_\infty \frac{\delta_T}{\delta L} + u_\infty \frac{\delta_T^2}{\delta L} = \frac{\alpha}{\delta_T^2}$$

$$u_\infty \frac{\delta_T}{\delta L}(1 + \delta_T) =$$

Skip it!

$$\frac{\delta_T}{L} \sim Pr^{-1/3} Re_L^{-1/2}$$

$$\boxed{\frac{\delta}{L} \sim Re_L^{-1/2}, \quad L \sim \frac{\delta}{Re_L^{-1/2}}}$$

Form ~~...~~ T considerations

$$\cancel{\frac{\delta_T}{\delta Re_L^{-1/2}}}$$

$\frac{\delta_T}{\delta} \sim Pr^{-1/3}$, but assumed $\delta >> \delta_T$

$$Pr^{-1/3} << 1 \text{ or } Pr^{1/3} >> 1$$

(11) 11/17/08

Recall B.L. Problem

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

~~Boundary Conditions~~

Non-dimensionalize:

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L}$$

$$u_* = \frac{u}{U_\infty} \qquad v_* = \frac{v}{U_\infty}$$

$$T^* = \frac{T - T_s}{T_\infty - T_s}$$

$\boxed{\text{Eq. 1}} \Rightarrow$ $$\cancel{\frac{U_\infty}{L}}\frac{\partial u_*}{\partial x_*} + \cancel{\frac{U_\infty}{L}}\frac{\partial v_*}{\partial y_*} = 0$$

$$\boxed{\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} = 0} \qquad (4)$$

~~$\boxed{\text{EQ. 3}} \Rightarrow$~~ $$\cancel{\frac{U_\infty u_*(T_\infty - T_s)}{L}\frac{\partial T_*}{\partial x_*}} +$$

$\boxed{\text{EQ. 2}}$

$$U_\infty u_* \frac{U_\infty}{\cancel{L}}\frac{\partial u_*}{\partial x_*} + \frac{U_\infty v_* U_\infty}{\cancel{L}}\frac{\partial u_*}{\partial y_*} = -\frac{1}{\rho}\frac{1}{\cancel{L}}\frac{\partial P_\infty}{\partial x_*} + \nu\frac{U_\infty}{\cancel{L^2}}\frac{\partial^2 u_*}{\partial y_*^2}$$

$$u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*} = \frac{-1}{\rho U_\infty^2}\frac{\partial P_\infty}{\partial x_*} + \frac{\nu}{U_\infty L}\frac{\partial^2 u_*}{\partial y_*^2}$$

$\frac{1}{Re_L}$

Define $p_* = \frac{P_\infty}{\rho U_\infty^2}$

$$\boxed{u_*\frac{\partial u_*}{\partial x_*} + v_*\frac{\partial u_*}{\partial y_*} = \frac{-\partial p_*}{\partial x_*} + \frac{1}{Re_L}\frac{\partial^2 u_*}{\partial y_*^2}} \qquad (5)$$

$\boxed{\text{EQ. 3}}$

$$\frac{u_\infty u_* \cancel{(T_\infty - T_s)}}{\cancel{L}} \frac{\partial T_*}{\partial x_*} + \frac{u_\infty v_* \cancel{(T_\infty - T_s)}}{\cancel{L}} \frac{\partial T_*}{\partial y_*} = \alpha \frac{\cancel{(T_s - T_\infty)}}{L^{\cancel{2}}} \frac{\partial^2 T_*}{\partial y_*^2}$$

$$u_* \frac{\partial T_*}{\partial x_*} + v_* \frac{\partial T_*}{\partial y_*} = \frac{\alpha}{u_\infty L} \frac{\partial^2 T_*}{\partial y_*^2}$$

$$\frac{1}{\frac{\alpha}{\nu}\;\frac{\nu}{u_\infty L}} \qquad \frac{\alpha}{\nu} \to Pr \qquad \frac{\nu}{u_\infty L} \to Re_L$$

Summarize:

SOLVE:

$$\boxed{\begin{aligned} &\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} = 0 \\ &u_* \frac{\partial u_*}{\partial x_*} + v_* \frac{\partial u_*}{\partial y_*} = -\frac{\partial p_*}{\partial x_*} + \frac{1}{Re_L} \frac{\partial^2 u_*}{\partial y_*^2} \\ &u_* \frac{\partial T_*}{\partial x_*} + v_* \frac{\partial T_*}{\partial y_*} = \frac{1}{Re_L Pr} \frac{\partial^2 T_*}{\partial y_*^2} \end{aligned}}$$

SUBJECT TO:

1. $u_*(x_*, 0) = 0$ non-slip
2. $v_*(x_*, 0) = 0$ impermeable
3. $u_*(x_*, \infty) = 1$
4. $T_*(x_*, 0) = 0$
5. $T_*(x_*, \infty) = 1$

NOT used here (don't need v_*)

GET:

$u_* = f(x_*, y_*, p_*, Re_L)$

$T_* = f(x_*, y_*, p_*, Re_L, Pr)$

since $T_* = f(u_*)$

Similarity Params (see p 370 in text)

$$\boxed{\begin{aligned} Bi &= \frac{hL}{k} \\ C_f &= \frac{\tau_s}{\frac{1}{2}\rho V^2} \\ Nu &= \frac{hL}{k_f} \\ Re &= \frac{uL}{\nu} \\ Pr &= \frac{\nu}{\alpha} \end{aligned}}$$

Important ones here