Including Turbulent Fluctuations in Navier-Stokes

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ENGR 3443 Thermal Fluid Engineering II

Momentum Eq. (Navier-Stokes) NS

$$\rho \frac{d\vec{V}}{dt} = -\vec{\nabla} p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

where:

$$\underbrace{\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{Substantial derivative}}$$

$\boxed{\text{x-component of NS}}$

$$\rho \Big[\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\frac{du}{dt}}\Big] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Note: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ } continuity

Book suggests that

$$\cancel{\frac{du}{dt} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(w^2)}$$

$$\frac{du}{dt} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)$$

$$= 2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial w}{\partial z} + w\frac{\partial u}{\partial z}$$

$$= u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

QED

$$\frac{du}{dt} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)$$

At Steady-State

SS, x-comp. of N.S. then becomes

Eq. (1)

$$\underbrace{\rho\left[\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right]}_{\text{LHS}} = \underbrace{-\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]}_{\text{RHS}}$$

For turbulent flow:

$$\text{Mean vel.} = \bar{u} = \frac{1}{T}\int_0^T u\,dt$$

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w'$$

Turbulent Fluctuations

LHS of Eq. (1) becomes:

$$\underbrace{\rho\left[\frac{\partial}{\partial x}(\bar{u}+u')^2 + \frac{\partial}{\partial y}(\bar{u}+u')(\bar{v}+v') + \frac{\partial}{\partial z}(\bar{u}+u')(\bar{w}+w')\right]}$$

$$= \frac{\partial}{\partial x}(\bar{u}^2 + 2\bar{u}u' + u'^2) + \frac{\partial}{\partial y}(\bar{u}\bar{v} + \bar{u}v' + u'\bar{v} + u'v')$$

$$+ \frac{\partial}{\partial z}(\bar{u}\bar{w} + \bar{u}w' + u'\bar{w} + u'w')$$

Note:

① $\overline{u'} = \frac{1}{T}\int_0^T u'\,dt = 0$ (u' takes on +/- values that tend to cancel if T is large)

$\overline{v'} = 0$, $\overline{w'} = 0$ Also

② $\overline{\bar{u}\bar{v}} = \frac{1}{T}\int_0^T \bar{u}\bar{v}\,dt = \frac{\bar{u}\bar{v}}{T}\int_0^T dt = \bar{u}\bar{v}$

$\overline{\bar{u}\bar{w}} = \bar{u}\bar{w}$, $\overline{\bar{u}\bar{u}} = \bar{u}\bar{u} = \bar{u}^2$ Also

③ $\overline{\bar{u}v'} = \frac{1}{T}\int_0^T \bar{u}v'\,dt = \frac{\bar{u}}{T}\cancelto{0}{\int_0^T v'\,dt} = 0$

$\overline{\bar{u}u'} = 0$, $\overline{\bar{u}w'} = 0$ Also

$$\frac{1}{T}\int_0^T [\text{LHS of Eq (1)}]\,dt$$

$$= \rho\left[\frac{\partial}{\partial x}(\bar{u}^2 + \overline{u'^2}) + \frac{\partial}{\partial y}(\bar{u}\bar{v} + \overline{u'v'}) + \frac{\partial}{\partial z}(\bar{u}\bar{w} \quad \overline{u'w'})\right]$$

$$= \rho\left[\frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\bar{u}\bar{v}) + \frac{\partial}{\partial z}(\bar{u}\bar{w})\right]$$

$$+ \rho\left[\frac{\partial}{\partial x}(\overline{u'^2}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'})\right]$$

$$= \rho \underbrace{\frac{d\bar{u}}{dt}}_{\text{substantial derivative}} + \rho\left[\frac{\partial}{\partial x}(\overline{u'^2}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'})\right] \qquad \text{Eq. (2)}$$

RHS of Eq. (1) [Define $\bar{p} = \frac{1}{T}\int_0^T p\,dt$]

$$-\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Integrate (Average) over time ...

$\bar{u} + u'$ Eq. (3)

$$\underbrace{\frac{1}{T}\int_0^T -\frac{\partial p}{\partial x}\,dt}_{-\frac{\partial \bar{p}}{\partial x}} + \underbrace{\frac{1}{T}\int_0^T \rho g_x\,dt}_{\rho g_x} + \underbrace{\frac{1}{T}\int_0^T \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]dt}_{\mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial w^2}\right]}$$

Equate Eqs. (2 & 3)

$$\frac{1}{T}\int_0^T (\text{LHS of Eq. 1})\,dt = \frac{1}{T}\int_0^T (\text{RHS of Eq. 1})\,dt$$

$$\rho\frac{d\bar{u}}{dt} + \rho\left[\frac{\partial}{\partial x}(\overline{u'^2}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'})\right]$$

$$= -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right]$$

$$\boxed{\rho\frac{d\bar{u}}{dt} = -\frac{\partial \bar{p}}{\partial x} + \rho g_x + \frac{\partial}{\partial x}\left[\mu\frac{\partial \bar{u}}{\partial x} - \rho\overline{u'^2}\right] + \frac{\partial}{\partial y}\left[\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right] + \frac{\partial}{\partial z}\left[\mu\frac{\partial \bar{u}}{\partial z} - \rho\overline{u'w'}\right]}$$

This matches Eq 6.21 in text ←