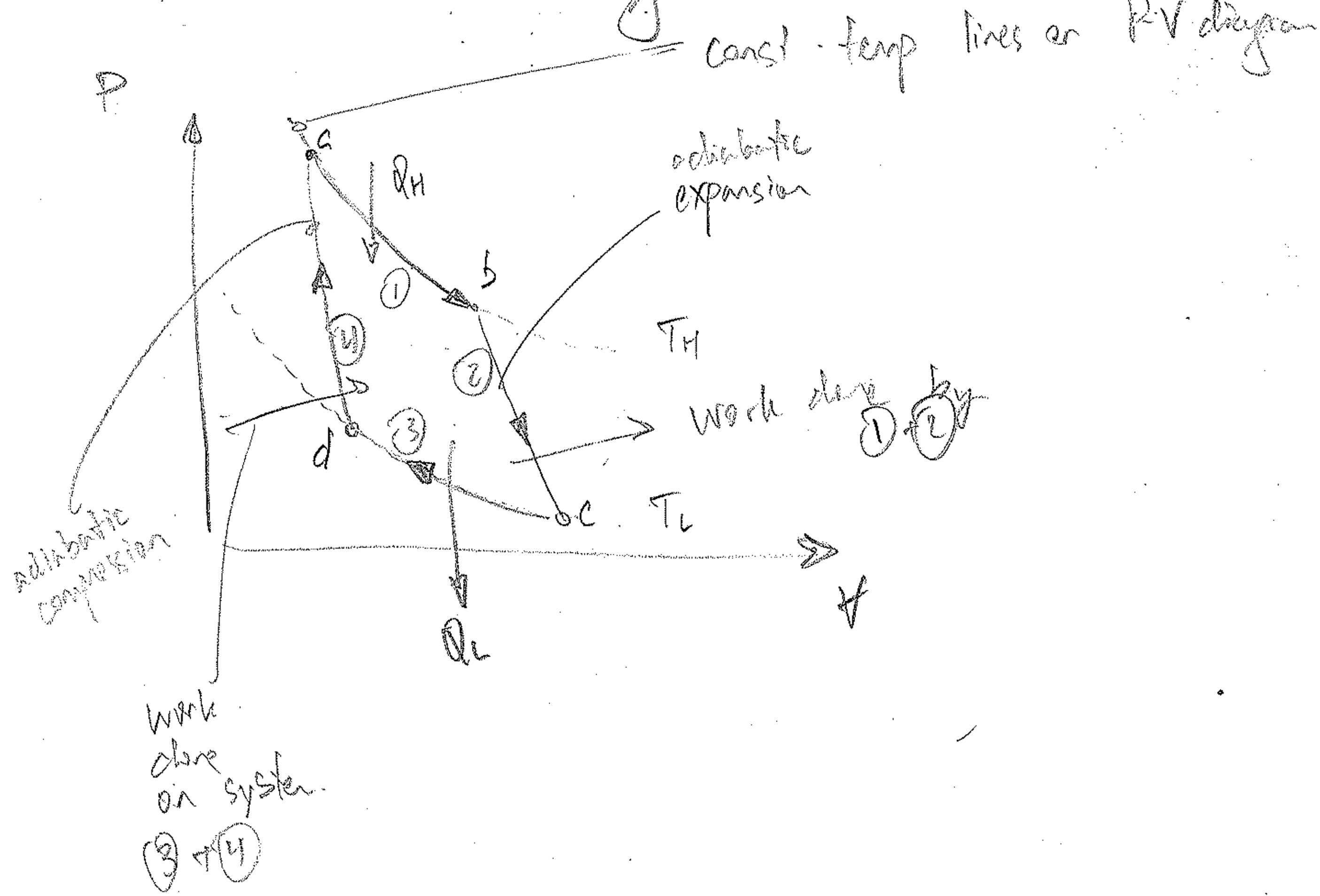
Show that $\eta_{th} = 1 - \frac{T_L}{T_H}$ for an ideal gas

CARNOT CYCLE w/ Ideal Gas

1. Heat Trans. @ const. temp. T_H
2. Expansion $T_H \to T_L$ (adiabatic)
3. Heat Trans @ const. temp. T_L
4. Compression $T_L \to T_H$ (adiabatic)

Let's draw a P-V diagram



First Law along 1 [a → b]

$$Q_H = \cancelto{0\ [\text{const. temp}]}{\Delta U} + {}_aW_b = \int_a^b p\, d\forall$$

$$p\forall = mRT \qquad \text{Ideal Gas}$$

$$p = \frac{mRT}{\forall}$$

$$Q_H = \int_a^b \frac{mRT_H}{\forall}\, d\forall = mRT_H \ln\frac{\forall_b}{\forall_a}$$

First Law along 3 [c → d]

$$Q_L = \int_c^d \frac{mRT_L}{\forall}\, d\forall = mRT_L \ln\frac{\forall_d}{\forall_c}$$

← this will be <u>negative</u>

$$|Q_L| = mRT_L \ln\frac{\forall_c}{\forall_d}$$

<u>Examine</u>

$$\frac{|Q_L|}{Q_H} = \frac{T_L \ln\frac{\forall_c}{\forall_d}}{T_H \ln\frac{\forall_b}{\forall_a}}$$

<u>Consider</u>

$$\left.\begin{aligned} p_a\forall_a &= mRT_H \\ p_b\forall_b &= mRT_H \end{aligned}\right\} \quad p_a\forall_a = p_b\forall_b$$

$$\left.\begin{aligned} p_c \forall_c &= mR\,T_L \\ p_d \forall_d &= mR\,T_L \end{aligned}\right\} \quad p_c\forall_c = p_d \forall_d$$

So,

$$\frac{\forall_b}{\forall_a} = \frac{p_a}{p_b} \quad \& \quad \frac{\forall_d}{\forall_c} = \frac{p_c}{p_d}$$

<u>Consider adiabatic processes w/ ideal gases</u> (i.e. along $b \to c$)

$$\cancel{Q}^{\,0} = \Delta U + W$$

$$\Delta U = -W = -\int_b^c p\, d\forall$$

$$\downarrow$$

$$m \int_{T_H}^{T_L} C_{vo}\, dT = -\int_b^c p\, d\forall$$

~~Assume $C_{vo} \approx$ const.~~

$$\cancel{mC_{vo}(T_L - T_H) + \int_b^c p\, d\forall = 0}$$

$$p\forall = mRT$$

$$\forall\, dp + p\, d\forall = mR\, dT$$

$$p\, d\forall = mR\, dT - \forall\, dp$$

$$m C_{v0} dT = -p d\forall$$

$$m C_{v0} dT + p d\forall = 0$$

$$m C_{v0} dT + m R dT - \forall dp = 0$$

$$m C_{v0} dT + m(C_{p0} - C_{v0}) dT - \forall dp = 0$$

$$m dT \left[\cancel{C_{v0}} + C_{p0} - \cancel{C_{v0}} \right] - \forall dp = 0$$

<u>Note</u> $$dT = \frac{p d\forall + \forall dp}{mR}$$ To get rid of 'm'

So,

$$\frac{C_{p0}}{R} \left[p d\forall + \forall dp \right] - \forall dp = 0$$

$$\frac{C_{p0}}{R} p d\forall + \left[\frac{C_{p0}}{R} - 1 \right] \forall dp = 0$$

$$\frac{C_{p0}}{C_{p0} - C_{v0}}$$

$$\frac{1}{1 - \frac{1}{\gamma}} = \frac{\gamma}{\gamma - 1}$$

$$\gamma = \frac{C_{p0}}{C_{v0}}$$

$$\frac{\gamma}{\gamma - 1} p d\forall + \frac{1}{\gamma - 1} \forall dp = 0$$

$$\frac{\gamma}{\gamma - 1} - 1 = \frac{\gamma - \gamma + 1}{\gamma - 1} = \frac{1}{\gamma - 1}$$

$$p = \frac{mRT}{\forall}$$

$$\forall = \frac{mRT}{p}$$

Divide by $p\forall$ ...

$$\int \left(\gamma \frac{d\forall}{\forall} + \frac{dp}{p} \right) = \int 0 \quad = C = \text{constant}$$

$$\gamma \ln \forall + \ln p = C$$

$$\ln \forall^{\gamma} + \ln p = C$$

$$\ln [p \forall^{\gamma}] = C$$

$$p \forall^{\gamma} = e^{C} = \text{constant}$$

So adiabatic expansion or compression of ideal gas is polytropic w/ $\boxed{n = \gamma = \frac{c_p}{c_v}}$

So, $p_b V_b^{\gamma} = p_c V_c^{\gamma}$ b-c

$p_d V_d^{\gamma} = p_a V_a^{\gamma}$ d-a

$$V_d = \left(\frac{p_a}{p_d}\right)^{1/\gamma} V_a$$

$$V_c = \left(\frac{p_b}{p_c}\right)^{1/\gamma} V_b$$

$$\frac{V_d}{V_c} = \left(\frac{p_a p_c}{p_d p_b}\right)^{1/\gamma} \frac{V_a}{V_b}$$

$$\boxed{\begin{aligned} \frac{p_a}{p_b} &= \frac{V_b}{V_a} \\ \frac{p_c}{p_d} &= \frac{V_d}{V_c} \end{aligned}}$$

FROM EARLIER

$$\frac{V_d}{V_c} = \left(\frac{V_b}{V_a}\right)^{1/\gamma} \left(\frac{V_d}{V_c}\right)^{1/\gamma} \frac{V_a}{V_b}$$

$$\left(\frac{V_d}{V_c}\right)^{1-1/\gamma} = \left(\frac{V_b}{V_a}\right)^{1/\gamma - 1}$$

$\frac{\gamma-1}{\gamma}$

$\frac{1-\gamma}{\gamma} = \left(\frac{\gamma}{1-\gamma}\right)^{-1}$

$$\frac{V_d}{V_c} = \left[\frac{V_b}{V_a}\right]^{-1} = \frac{V_a}{V_b}$$

$$\frac{V_c}{V_d} = \frac{V_b}{V_a}$$

NOTE units of T must be absolute here

Finally:

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H} \cancel{\frac{\ln\left[\frac{V_b}{V_a}\right]}{\ln\left(\frac{V_b}{V_a}\right]}}$$

So,

$$1 - \frac{Q_L}{Q_H} \quad \text{any cycle}$$

$$\eta_{th}^{CARNOT} = 1 - \frac{T_L}{T_H} \quad \text{CARNOT CYCLE}$$

This all works because abs. temp. scale is defined such that

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H} \text{ for } \underline{\text{all}}$$

Carnot Cycles