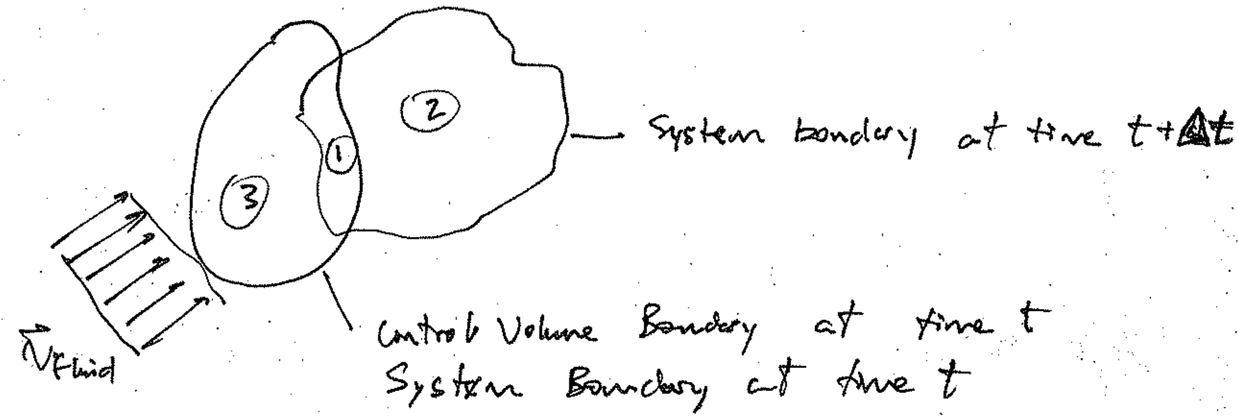
THE CONTINUITY EQUATION

(this applies to a "system")

Basically this is conservation of mass... but we need to learn to apply this (and other things) to a "control volume" → Watch the same volume in space over time.



Mass of the system is const.

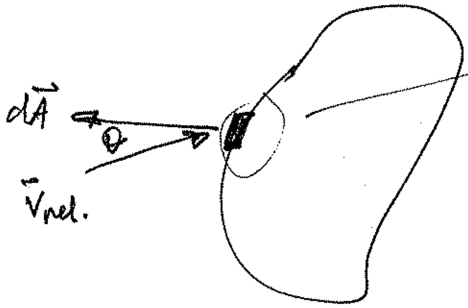
$$m_1(t) + m_3(t) = m_1(t+\Delta t) + m_2(t+\Delta t)$$

$$\lim_{\Delta t \to 0} \left\{ \underbrace{\frac{m_1(t+\Delta t) - m_1(t)}{\Delta t}}_{\text{rate mass is stored in C.V.}} = \cancel{\frac{m_3(t) - m_2(t+\Delta t)}{\Delta t}} \right\}$$

$$\underbrace{\frac{m_3(t)}{\Delta t}}_{\text{rate mass enters the C.V.}} - \underbrace{\frac{m_2(t+\Delta t)}{\Delta t}}_{\text{rate mass leaves the C.V.}}$$

$$\sum \dot{m}_{stored} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

FOR DETAILED CALCULATIONS OF MASS FLOW RATE CONSIDER...

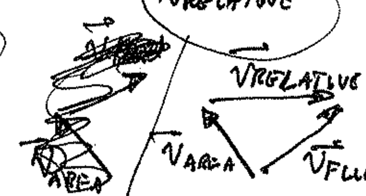


① In general area can be moving ...

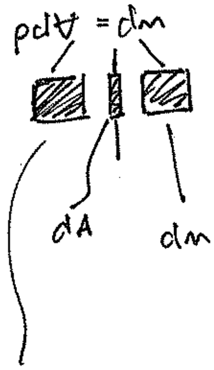
② Fluid is also moving

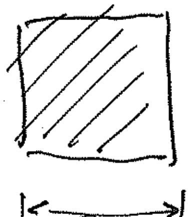
$$\vec{V}_{RELATIVE} = \vec{V}_{fluid} - \vec{V}_{AREA}$$

③



"Fluid vel. relative to C.V. dA"





$(V_n)_{rel.}\, dt = dl$

normal component of relative fluid vel.

So $d\forall = dl\, dA = (V_n)_{rel.}\, dt\, dA$

$V_{rel.} \cos\theta$

So... amount of mass entering is $dm = \rho d\forall$

$= \rho (V_{rel} \cos\theta)\, dt\, dA$

rate mass enters 'dA' is

$$\left.\frac{dm}{dt}\right|_{dA} = \rho V_{rel} \cos\theta\, dA$$

$$= \rho \vec{V}_{rel} \cdot d\vec{A}$$

Note $d\vec{A}$ is unit outward normal

$\vec{V}_{rel.} \cdot d\vec{A} = V_{rel}\, dA \cos\theta$

Note... fluid going out +

" going in −

So... to get the net-rate fluid leaves the C.V.

$$\cancel{\dot{m}_{out}} \Rightarrow \int_A \rho \vec{V}_{rel.} \cdot d\vec{A} = \Sigma \dot{m}_{out} - \Sigma \dot{m}_{in}$$

Thus, $\Sigma \dot{m}_{stored} = -\int_A \rho \vec{V}_{rel.} \cdot d\vec{A}$, $\Sigma \dot{m}_{stored} = \frac{\partial}{\partial t}\left[\int_V \rho d\forall\right]$

$$\frac{\partial}{\partial t}\left[\int_V \rho d\forall\right] = -\int_A \rho \vec{V}_{rel.} \cdot d\vec{A}$$

Note, the ~~D~~ivergence theorem... $\int_A \vec{\varphi} \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{\varphi} \, d\forall$

Now let $\vec{\varphi} = \rho \vec{V}_{rel.}$

A is ~~an~~ area enclosing $\forall$
$\vec{\varphi}$ is a vector

So...

$$\frac{\partial}{\partial t}\left(\int_\forall \rho d\forall\right) = -\int_V \vec{\nabla} \cdot (\rho \vec{V}_{rel}) d\forall$$

$$\frac{\partial}{\partial t}\left(\int_\forall \rho d\forall\right) + \int_\forall \vec{\nabla} \cdot (\rho \vec{V}_{rel}) d\forall = 0$$

general (e.g. C.V. can change shape w/ time.)

If you choose ~~if~~ the C.V stays the same!

$$\int_\forall \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}_{rel})\right] d\forall = 0$$

For an arbitrary C.V. ... to make the integral always go to zero ... the integrand must be zero!!

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}_{rel}) = 0$$

If ~~the~~ C.V. is motionless then $\vec{V}_{rel} = \vec{V}_{fluid}$ ..

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$ $\boxed{\text{CONTINUITY EQUATION}}$

For CARTESIAN COORDS. $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

FOR $\rho = $ const.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \vec{\nabla} \cdot \vec{V} = 0$$

INCOMPRESSIBLE FLUID

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$$

$$\underbrace{\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right)}_{\text{substantial derivative ... ("convective derivative")}} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{D\rho}{Dt}$$

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0$$

SAME AS

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$$

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underbrace{\vec{\nabla}\cdot(\rho\vec{V}) - \rho\vec{\nabla}\cdot\vec{V}}_{\text{WHAT IS THIS?}}$$

$$\vec{V}\cdot\vec{\nabla}\rho = (u\hat{i} + v\hat{j} + w\hat{k})\cdot\left(\frac{\partial \rho}{\partial x}\hat{i} + \frac{\partial \rho}{\partial y}\hat{j} + \frac{\partial \rho}{\partial z}\hat{k}\right)$$

$$= u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{V}\cdot\vec{\nabla}\rho$$

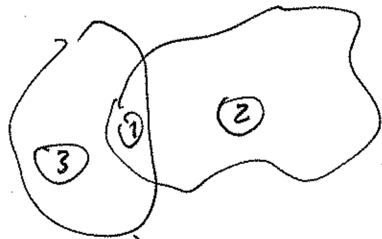
Nice compact def. $\frac{D\rho}{Dt}$

[relative if needed]

THE MOMENTUM EQUATION

For a fluid particle : $\underbrace{\vec{F}}_{\text{net external force on fluid particle (the system)}} = m\dfrac{d\vec{V}}{dt} = \dfrac{d(m\vec{V})}{dt}$ ← this is for essentially a system

Go BACK TO CONTROL VOLUME APPROACH



① + ② final system

① + ③ initial system

Initial Sys. Momentum $= m_1\vec{V}_1(t) + m_3\vec{V}_3(t)$

Final Sys. Momentum $= m_1\vec{V}_1(t+\Delta t) + m_2\vec{V}_2(t+\Delta t)$

$\underbrace{\Delta \text{Momentum}}_{\text{"for system"}} = \left(m_1\vec{V}_1(t+\Delta t) - m_1\vec{V}_1(t)\right) + m_2\vec{V}_2(t+\Delta t) - m_3\vec{V}_3(t)$

$$\vec{F} = \frac{d(m\vec{V})}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\text{Momentum}}{\Delta t}$$

$$\vec{F} = \underbrace{\frac{m\vec{V}_1(t+dt) - m_1\vec{V}_1(t)}{dt}}_{\text{rate of mom. storage in C.V.}} + \underbrace{\frac{m_2\vec{V}_2(t+dt)}{dt}}_{\text{rate momentum flows out}} - \underbrace{\frac{m_3\vec{V}_3(t)}{dt}}_{\text{rate momentum flows in}}$$

$$\vec{F} = \dot{\text{mom.}}_{\text{stored}} + \dot{\text{momentum}}_{\text{out}} - \dot{\text{momentum}}_{\text{in}}$$

$\rho\vec{V}\,d\forall$ momentum contained in $d\forall$ at time t

$\int_{\forall}\rho\vec{V}\,d\forall$ " " " $\forall$ (c.v.) " " "

$\frac{d}{dt}\int_{\forall}\rho\vec{V}\,d\forall$ ~ rate of change of momentum in C.V.

~~Mom~~ momentum stored $= \frac{d}{dt}\int_{\forall} \rho\vec{V}\,d\forall$

Recall, the differential mass that enters the C.V. in time dt is

$$\rho\vec{V}_{rel}\cdot d\vec{A}\,dt \qquad \begin{matrix}\text{neg. means } \underline{\text{in}} \\ \text{pos. means } \underline{\text{out}}\end{matrix}$$

The velocity of the fluid entering (relative to a stationary observer) is $\vec{V}$

$$\frac{(\rho\vec{V}_{rel}\cdot d\vec{A}\,\cancel{dt})\vec{V}}{\cancel{dt}} = (\rho\vec{V}_{rel}\cdot d\vec{A})\vec{V}$$

rate momentum crosses C.V. boundary

$\int_A \vec{V}\rho\vec{V}_{rel}\cdot d\vec{A}$ — net rate that momentum leaves C.V.

same as momentum out − momentum in

So... $\vec{F} = \frac{d}{dt}\int_{\forall}\rho\vec{V}\,d\forall + \int_A \vec{V}\rho\vec{V}_{rel}\cdot d\vec{A}$

VECTOR IDENTITY: $\int_A \vec{Q}(\vec{R}\cdot d\vec{A}) = \int_{\forall}[\vec{Q}\,\vec{\nabla}\cdot\vec{R} + (\vec{R}\cdot\vec{\nabla})\vec{Q}]\,d\forall$

"transform surface area integral to volume"

$\vec{Q} = \vec{V}, \quad \vec{R} = \rho\vec{V}_{rel}$

$$\vec{F} = \frac{d}{dt}\int_{\forall}\rho\vec{V}\,d\forall + \int_{\forall}[\vec{V}\,\vec{\nabla}\cdot(\rho\vec{V}_{rel}) + (\rho\vec{V}_{rel}\cdot\vec{\nabla})\vec{V}]\,d\forall$$

If C.V. has fixed size ... $\frac{d}{dt}\int_{\forall}\rho\vec{V}\,d\forall = \int_{\forall}\frac{\partial}{\partial t}(\rho\vec{V})\,d\forall$, AND

$$\vec{F} = \int_{\forall}\left[\frac{\partial(\rho\vec{V})}{\partial t} + \vec{V}\,\vec{\nabla}\cdot(\rho\vec{V}_{rel}) + (\rho\vec{V}_{rel}\cdot\vec{\nabla})\vec{V}\right]d\forall$$

For infinitesimal C.V. ($d\forall$ in size)

$$\frac{\vec{F}}{d\forall} = \frac{\partial(\rho\vec{V})}{\partial t} + \vec{V}\,\vec{\nabla}\cdot(\rho\vec{V}_{rel}) + (\rho\vec{V}_{rel}\cdot\vec{\nabla})\vec{V}$$

$$\vec{V}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\vec{V}}{\partial t}$$

Rewrite as:

$$\frac{\vec{F}}{d\forall} = \vec{V}\underbrace{\left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}_{rel})\right]}_{\text{continuity eq. } = 0!} + \rho\underbrace{\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}_{rel}\cdot\vec{\nabla})\vec{V}\right]}_{\text{general form of substantial derivative}}$$

$$\frac{\partial (X)}{\partial t} + (\vec{V}_{rel}\cdot\vec{\nabla})(X)$$

$$\frac{D(X)}{Dt}$$

So...

$$\underbrace{\frac{\vec{F}}{d\forall}}_{\frac{\text{force}}{\text{volume}}} = \underbrace{\rho\frac{D\vec{V}}{Dt}}_{\frac{m\vec{a}}{\text{volume}}}$$

"total derivative of fluid particle velocity" [Total Acceleration] (note, as observed by moving particle)

$$\frac{\partial \vec{V}}{\partial t} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}}$$

FOR A STATIONARY C.V. :

$$\frac{\vec{F}}{d\forall} = \rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}_0\cdot\vec{\nabla})\vec{V}\right]$$

IN Rect. Coords:

$$\frac{F_x}{d\forall} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

$\frac{F_y}{d\forall}$ & $\frac{F_z}{d\forall}$ by analogy

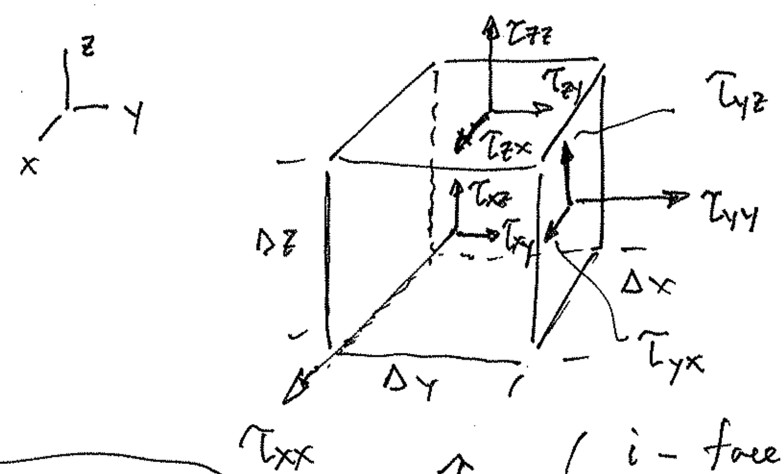
Surface & volume forces
(contact) (body)

$\underbrace{\text{(contact)}}_{\text{shear \& pressure}}$ $\underbrace{\text{(body)}}_{\text{gravity \& electromagnetic}}$

$\downarrow$

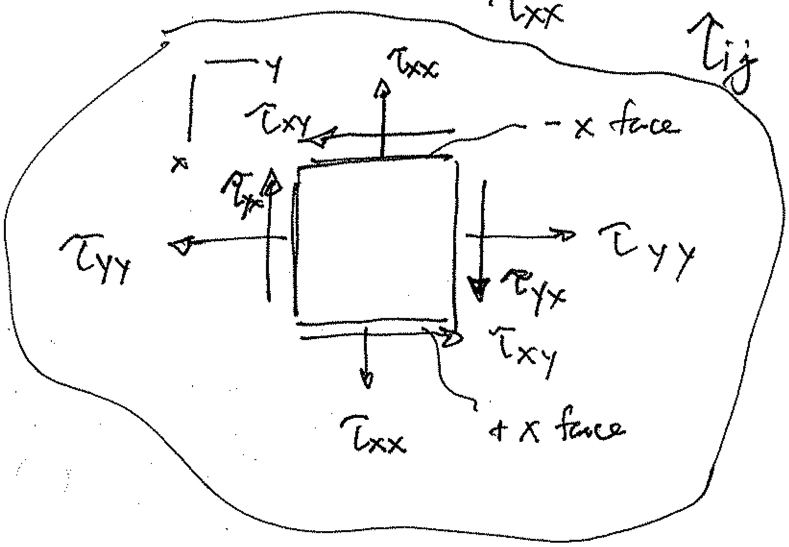
Pretty EASY (sic) to ANALYZE . . Consider . . .

$$\tau = \frac{\text{Force}}{\text{Area}} = \text{stress}$$



NOTE: Other 3 faces have stresses too

τ_{ij} { i – face it's on (normal direction x, y, z)
j – direction (x, y, z)



Sign convention … all are positive as shown

So let's add up total force in x-direction …

NOTE: $B_x \equiv$ Body Force / Volume $= \dfrac{\text{Body Force}}{\Delta x \Delta y \Delta z}$

NOTE: center of cube above is x, y, z

Positive x-face: $+ \cancel{\tau_{yx}|_{x,y+\frac{\Delta y}{2},z}(\Delta x \Delta z)}\quad \tau_{xx}|_{x+\frac{\Delta x}{2},y,z}(\Delta y \Delta z)$

" y-face: $\tau_{yx}|_{x,y+\frac{\Delta y}{2},z}(\Delta x \Delta z)$

" z-face: $\tau_{zx}|_{x,y,z+\Delta z/2}(\Delta x \Delta y)$

Neg. x-face: $-\tau_{xx}|_{x-\frac{\Delta x}{2},y,z}(\Delta y \Delta z)$

Neg. y-face: $-\tau_{yx}|_{x,y-\frac{\Delta y}{2},z}(\Delta x \Delta z)$

Neg. z-face: $-\tau_{zx}|_{x,y,z-\Delta z/2}(\Delta x \Delta y)$

Note, all of the surface forces are evaluated at $t+\Delta t/2$

So... $F_x = \left(\tau_{xx}\big|_{x+\frac{\Delta x}{2},y,z} - \tau_{xx}\big|_{x-\frac{\Delta x}{2},y,z} \right)(\Delta y \Delta z)$

$$+ \left(\tau_{yx}\big|_{x,y+\frac{\Delta y}{2},z} - \tau_{yx}\big|_{x,y-\frac{\Delta y}{2},z} \right)(\Delta x \Delta z)$$

$$+ \left(\tau_{zx}\big|_{x,y,z+\Delta z/2} - \tau_{zx}\big|_{x,y,z-\Delta z/2} \right)(\Delta x \Delta y)$$

$$+ B_x\big|_{x,y,z} \; \Delta x \Delta y \Delta z$$

Divide by $\Delta x \Delta y \Delta z$, then take limit as $d\forall \to 0$ [i.e. $\Delta x \Delta y \Delta z \to 0$]

$$\frac{F_x}{d\forall} = B_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

So... $\boxed{B_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \frac{Du}{Dt}}$

can write down y, z by analogy

Generally need a way to relate τ to the velocity gradient...

For Newtonian Fluids: linear relationship

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \tau_{zx}$$

$$\tau_{xx} = -p + \mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)$$

$$\tau_{yy} = -p + \mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)$$

$$\tau_{zz} = -p + \mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)$$

Explain: See White §'s 2-4.2, 2-4.3

INVISCID FLUIDS

Note: $\tau_{ii} = -p + (\tau_{ii})_{viscous}$, If $\mu = 0$, $\tau_{ii} = -p$

& $p = -\dfrac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3}$

$$\rho\frac{Du}{Dt} = B_x + \frac{\partial}{\partial x}\left[-p + \mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

For μ = const.

$$\rho\frac{Du}{Dt} = B_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}\left[\vec{\nabla}\cdot\vec{V}\right] + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial y\partial x} + \mu\frac{\partial^2 w}{\partial z\partial x} + \mu\frac{\partial^2 u}{\partial z^2}$$

$$= B_x - \frac{\partial p}{\partial x} + \mu\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\nabla^2 u} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right) - \frac{2}{3}\mu\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{V})$$

$$\frac{\partial}{\partial x}\underbrace{\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]}_{\vec{\nabla}\cdot\vec{V}}$$

$$\boxed{\rho\frac{Du}{Dt} = B_x - \frac{\partial P}{\partial x} + \mu\nabla^2 u + \frac{1}{3}\mu\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{V})}$$

The y & z equations would be ...

$$\rho \frac{Dv}{Dt} = B_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v + \frac{1}{3}\mu \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{V})$$

$$\rho \frac{Dw}{Dt} = B_z - \frac{\partial p}{\partial z} + \mu \nabla^2 w + \frac{1}{3}\mu \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{V})$$

<u>In Vector Form:</u>

$$\rho \frac{D\vec{V}}{Dt} = \vec{B} - \vec{\nabla}p + \mu\left[\nabla^2\vec{V} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{V})\right]$$

If ρ = const., then Continuity Eq. is

$$\frac{D\rho}{Dt} \quad \cancel{\frac{\partial \rho}{\partial t}}^{\,\to 0} + {}^{\rho}\vec{\nabla}\cdot(\rho\vec{V}) = 0$$

$$\vec{\nabla}\cdot\vec{V} = 0$$

So...

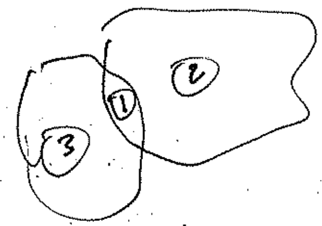
$$\boxed{\rho \frac{D\vec{V}}{Dt} = \vec{B} - \vec{\nabla}p + \mu\nabla^2\vec{V}}$$

NAVIER-STOKES EQUATIONS
FOR INCOMPRESSIBLE FLOW

In a formal sense... Boundary & Initial Conditions applied to the above Eqs. can be solved for $\vec{V}$ & p distributions

THE ENERGY EQUATION

Again referring to C.V. diagram



For a system of particles... $Q_{in} = \Delta E + W_{out}$

internal energy

Mechanical, Thermal, etc...

$E = E_{kinetic} + \cancel{E_{potential}}_{\text{(body force)}} + \cancel{E_{surface\ tension}} + \cancel{E_{EM\ applied}} + E_{thermal} + \ldots$

ignore ignore

include in W_{out}

$$\boxed{E = E_{kinetic} + E_{thermal}}$$

$$E_{initial} = E_1(t) + E_3(t) \qquad E_{final} = E_1(t+\Delta t) + E_2(t+\Delta t)$$

$$\Delta E = \left[E_1(t+\Delta t) - E_1(t)\right] + \left[E_2(t+\Delta t) - E_3(t)\right]$$

$$\frac{Q_{in}}{\Delta t} = \frac{E_1(t+\Delta t) - E_1(t)}{\Delta t} + \frac{E_2(t+\Delta t)}{\Delta t} - \frac{E_3(t)}{\Delta t} + \frac{W_{out}}{\Delta t}$$

TAKE LIMIT AS $\Delta t \to 0$ ...

$$\frac{d}{dt}(Q_{in}) - \frac{d}{dt}(W_{out}) = \frac{d}{dt}(E_{stored}) + \frac{d}{dt}(E_{out}) - \frac{d}{dt}(E_{in})$$

Define specific internal energy --- $e \equiv$ Energy/mass

Total energy in C.V. @ time $t = \int_{V} e\rho\, dV$

~~Total~~ Net rate energy leaves C.V. $= \dot{E}_{out} - \dot{E}_{in} = \int_A e(\rho \vec{V}_{rel} \cdot d\vec{A})$

$$\dot{Q}_{in} - \dot{W}_{out} = \frac{d\left(\int_{\forall} e\rho\, d\forall\right)}{dt} + \int_A e\left(\rho \vec{V}_{rel}\cdot d\vec{A}\right)$$

$\Downarrow$ DIVERGENCE THEOREM

$$\left\{ \int_A \vec{Q}\cdot d\vec{A} = \int_{\forall} \vec{\nabla}\cdot\vec{Q}\, d\forall \right.$$

$$\dot{Q}_{in} - \dot{W}_{out} = \frac{d}{dt}\left(\int_{\forall} e\rho\, d\forall\right) + \int_{\forall} \vec{\nabla}\cdot\left(e\rho\vec{V}_{rel}\right) d\forall$$

For C.V. fixed size & shape.-- $\frac{d}{dt}\int_{\forall} e\rho\, d\forall = \int_{\forall} \frac{\partial(e\rho)}{\partial t} d\forall$

$$\dot{Q}_{in} - \dot{W}_{out} = \int_{\forall}\left[\frac{\partial(e\rho)}{\partial t} + \vec{\nabla}\cdot\left(\cancel{\rho} e\rho\vec{V}_{rel}\right)\right] d\forall$$

$\boxed{\text{As } \forall \to d\forall}$

$$\frac{\dot{Q}_{in}}{d\forall} - \frac{\dot{W}_{out}}{d\forall} = \frac{\partial(e\rho)}{\partial t} + \vec{\nabla}\cdot\left(e\rho\vec{V}_{rel}\right)$$

Note: vector identity $\vec{\nabla}\cdot(\varphi\vec{F}) = (\vec{\nabla}\varphi)\cdot\vec{F} + \varphi\vec{\nabla}\cdot\vec{F}$

Note ...

$$e\frac{\partial\rho}{\partial t} + \rho\frac{\partial e}{\partial t} + \cancel{\vec{\nabla}(e\rho)\cdot\vec{V}_{rel} + (e\rho)\vec{\nabla}\cdot\vec{V}_{rel}}$$

$$+ \vec{\nabla}(e)\cdot\left(\rho\vec{V}_{rel}\right) + (e)\vec{\nabla}\cdot\left(\rho\vec{V}_{rel}\right)$$

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$e\frac{\partial\rho}{\partial t} + \rho\frac{\partial e}{\partial t} + \rho\vec{V}_{rel}\cdot\vec{\nabla}e + e\vec{\nabla}\cdot\left(\rho\vec{V}_{rel}\right)$$

$$\cancel{\rho}\, e\underbrace{\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}_{rel}\right)\right]}_{\text{THE CONTINUITY EQ.} = 0\,!!} + \rho\underbrace{\left[\frac{\partial e}{\partial t} + \vec{V}_{rel}\cdot\vec{\nabla}e\right]}_{\frac{De}{Dt}}$$

$$\frac{\dot{Q}_{in}}{d\forall} - \frac{\dot{W}_{out}}{d\forall} = \rho\frac{De}{Dt}$$

Heat diffuses in & out & might be generated ---

$$\dot{Q}_{in} = -\int_A \vec{q}\cdot d\vec{A} + \int_{\forall} \dot{q}\, d\forall$$

heat generation rate / $d\forall$

$$\int_A \vec{q}\cdot d\vec{A} = \int_{\forall} \vec{\nabla}\cdot\vec{q}\, d\forall \quad \text{[The Divergence Theorem again]}$$

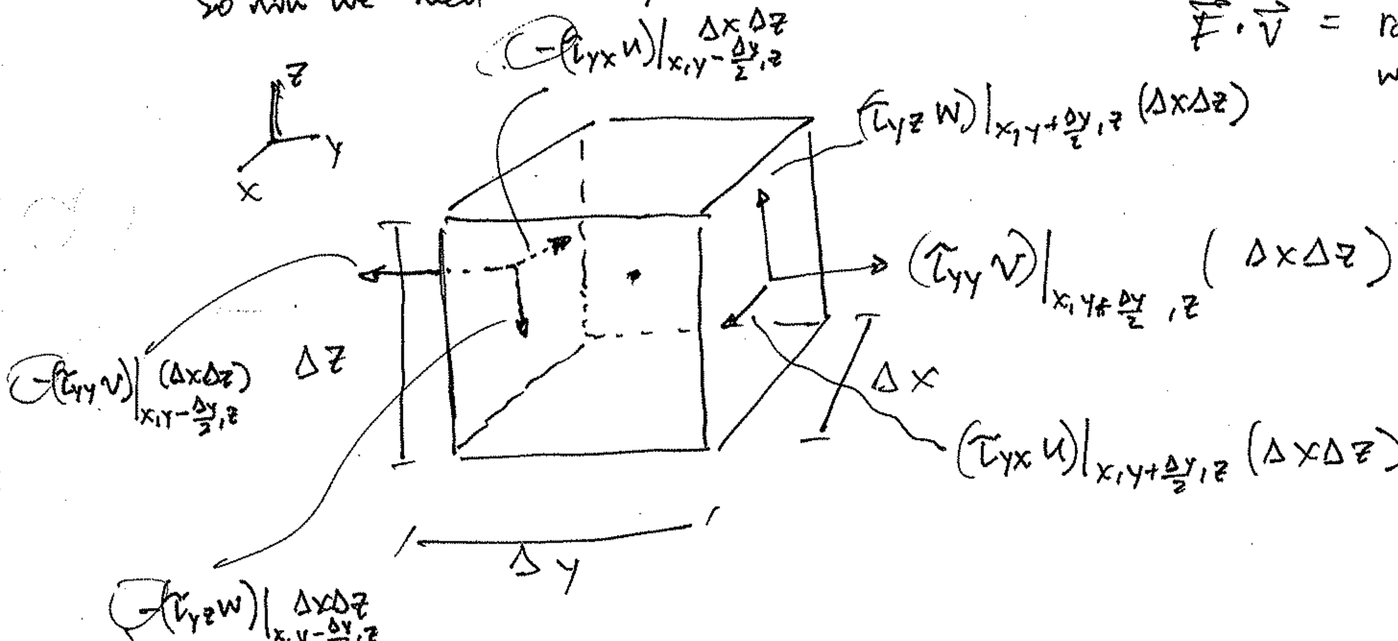
$$\dot{Q}_{in} = \int_{\forall} \left[\dot{q} - \vec{\nabla}\cdot\vec{q}\right] d\forall$$

$\boxed{\text{for } \forall \to d\forall}$

$$\frac{\dot{Q}_{in}}{d\forall} = \dot{q} - \vec{\nabla}\cdot\vec{q}$$

$$\rho\frac{De}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \frac{-\dot{W}_{out}}{d\forall}$$

So now we need $\dot{W}_{out}/d\forall$ ... consider [remember that $\vec{F}\cdot\vec{V}$ = rate of work done]



these negative signs are because τ & v directions are opposite

So for just the two (of six) sides shown...

$$\left[(\tau_{yy}v)|_{x,y+\frac{\Delta y}{2},z} - (\tau_{yy}v)|_{x,y-\frac{\Delta y}{2},z}\right]\Delta x\Delta z$$

$$+\left[(\tau_{yx}u)|_{x,y+\frac{\Delta y}{2},z} - (\tau_{yx}u)|_{x,y-\frac{\Delta y}{2},z}\right]\Delta x\Delta z$$

$$+\left[(\tau_{yz}w)|_{x,y+\frac{\Delta y}{2},z} - (\tau_{yz}w)|_{x,y-\frac{\Delta y}{2},z}\right]\Delta x\Delta z$$

Divide previous Eq. by $\Delta x \Delta y \Delta z$, let $\Delta x, \Delta y, \Delta z \to 0$

[now units are $\underline{\text{rate of work}}$ / volume]
$$\frac{\partial(\tau_{yy} v)}{\partial y} + \frac{\partial(\tau_{yx} u)}{\partial y} + \frac{\partial(\tau_{yz} w)}{\partial y}$$

If one goes through the other faces...

<u>x-faces</u>: $$\frac{\partial(\tau_{xx} u)}{\partial x} + \frac{\partial(\tau_{xy} v)}{\partial x} + \frac{\partial(\tau_{xz} w)}{\partial x}$$

<u>y-faces</u>: $$\frac{\partial(\tau_{zx} u)}{\partial z} + \frac{\partial(\tau_{zy} v)}{\partial z} + \frac{\partial(\tau_{zz} w)}{\partial z}$$

Note... {remember we said we would account for work done by body forces here...

$$\left.\begin{array}{l} B_x u \\ B_y v \\ B_z w \end{array}\right\} \text{Work done / Vol.}$$

note, this is what we have been doing (i.e. considering effects of outside forces)

<u>Finally</u>: $-\dot{W}_{out}/dV = \dot{W}_{in}/dV$

$$-\frac{\dot{W}_{out}}{dV} = \frac{\partial(\tau_{xx} u)}{\partial x} + \frac{\partial(\tau_{yx} u)}{\partial y} + \frac{\partial(\tau_{zx} u)}{\partial z} + B_x u$$
$$+ \frac{\partial(\tau_{xy} v)}{\partial x} + \frac{\partial(\tau_{yy} v)}{\partial y} + \frac{\partial(\tau_{zy} v)}{\partial z} + B_y v$$
$$+ \frac{\partial(\tau_{xz} w)}{\partial x} + \frac{\partial(\tau_{yz} w)}{\partial y} + \frac{\partial(\tau_{zz} w)}{\partial z} + B_z w$$

$$\frac{-\dot{W}_{out}}{d\forall} = u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + B_x\right) \rho\frac{Du}{Dt}$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + B_y\right) \rho\frac{Dv}{Dt}$$

$$+ w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + B_z\right) \rho\frac{Dw}{Dt}$$

$$+ \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$

$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$

$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

Introduce

$\frac{|\vec{V}|^2}{2} \equiv$ total K.E. per unit ~~of~~ mass $\quad |\vec{V}|^2 = u^2 + v^2 + w^2$

$$\frac{D}{Dt}\left(\frac{|\vec{V}|^2}{2}\right) = \frac{D(u^2/2)}{Dt} + \frac{D(v^2/2)}{Dt} + \frac{D(w^2/2)}{Dt}$$

$$= u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}$$

This appears in Eq. above

Let $I = e - |\vec{V}|^2/2$ This is the internal energy / mass due to temp. only ...

Recall

$$\rho\frac{De}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \frac{-\dot{W}_{out}}{d\forall}$$

first part of this is

$$\rho\frac{D}{Dt}\left(\frac{|\vec{V}|^2}{2}\right)$$

$$\rho\frac{DI}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \bigcirc$$

see next page...

$$\rho \frac{DI}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

RHS ...

recall specifications of τ_{ij} for a Newtonian Fluid

$$-p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\Phi$$

Φ ~ viscous dissipation $= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] - \frac{2}{3}(\vec{\nabla}\cdot\vec{V})^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2$

~~loss of energy~~
loss of mech. energy to thermal energy

$$\rho\frac{DI}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = -p\vec{\nabla}\cdot\vec{V} + \mu\Phi$$

Now... $H \equiv$ enthalpy is more convenient ---

$$h = I + pv = I + \frac{p}{\rho}$$

$$\frac{DI}{Dt} = \frac{Dh}{Dt} - \frac{D}{Dt}\left(\frac{p}{\rho}\right)$$

$$= \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

Continuity

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0$$

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{V}$$

$$= \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho}\vec{\nabla}\cdot\vec{V}$$

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} - \cancel{p\vec{\nabla}\cdot\vec{V}} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \cancel{-p\vec{\nabla}\cdot\vec{V}} + \mu\Phi$$

$$\rho \frac{Dh}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \frac{DP}{Dt} + \mu\Phi$$

still no temperature appears!

↳ through some thermo can show that

$$DH = C_p DT + \frac{1}{\rho}(1-\beta T)DP \qquad \beta = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{p=\text{const}}$$

thermal coeff. of exp.

$$\rho c_p \frac{DT}{Dt} + (\cancel{1}-\beta T)\frac{DP}{Dt} + \vec{\nabla}\cdot\vec{q} - \dot{q} = \frac{\cancel{DP}}{Dt} + \mu\Phi$$

$$\rho c_p \frac{DT}{Dt} = -\vec{\nabla}\cdot\vec{q} + \dot{q} + \beta T\frac{DP}{Dt} + \mu\Phi$$

Fourier's Law $\vec{q} = -k\vec{\nabla}T$

$$\rho c_p \frac{DT}{Dt} = -\vec{\nabla}\cdot(-k\vec{\nabla}T) + \dot{q} + \beta T\frac{DP}{Dt} + \mu\Phi$$

Assumptions -- (often made)

1. $k = \text{const.}$ (const. properties)
2. $\beta = \frac{1}{\rho}\frac{\partial\rho}{\partial T} = 0$ ($\rho = \text{const.}$ Incompressible)
3. $\Phi = 0$ ~ ~~almost always negl.~~ often negligible! (oil pipelines an exception)

under all three assumptions:

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \dot{q}$$

For example in Rectangular Coords --

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q}$$

$\rho c_p \frac{\partial T}{\partial t}$ $\quad \frac{kg}{m^3} \cdot \frac{J}{kg \cdot K} \cdot \frac{K}{s} = \frac{W}{m^3}$

LHS ≡ net rate of energy storage in C.V. / volume

RHS ≡ net rate of energy entering C.V. / volume

$$\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{\text{net rate of energy storage (stationary C.V.)}} + \underbrace{\rho c_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right]}_{\text{net rate of energy storage due to heat being carried in/out of C.V.}}$$

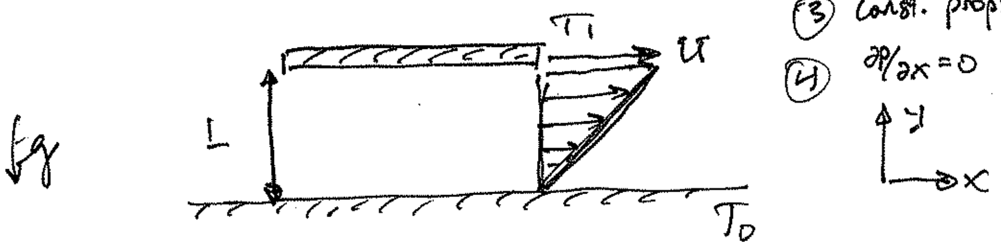
"convected"

$$k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$ — net rate of diffusion into C.V. / volume

~~Note, even if we are at steady-state~~ ...

Consider a case we can actually get a solution for ...

$\boxed{\text{COUETTE FLOW}}$



1. Steady
2. laminar flow
3. Const. props.
4. $\partial P / \partial x = 0$
5. $\frac{\partial ( )}{\partial z} = 0$ no edge effects
6. $\frac{\partial ( )}{\partial x} = 0$ no end effects
7. Newtonian Fluid

Boundary Conditions:

At $y = 0$, $u = 0$ (non-slip condition) $= w$ (also!)

$v = 0$ (impermeable boundary)

$T = T_0$.

At $y = L$, $u = U$, $w = 0$ (non-slip)

$v = 0$ (impermeable boundary)

$T = T_1$

Continuity: $\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \underset{(\rho = \text{const})}{\vec{\nabla}\cdot\vec{V} = 0} = \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0(6)} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0(5)} = 0$

$\frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{const.}$ (but already know $v = 0$)

Z-Momentum

$$\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{0(1)} + \underbrace{u\cancel{\frac{\partial w}{\partial x}}}_{0(6)} + \overbrace{\cancel{v}}^{0\text{ (continuity)}}\frac{\partial w}{\partial y} + \underbrace{w\cancel{\frac{\partial w}{\partial z}}}_{0(5)} = \frac{1}{\rho}\left[\overset{0}{\cancel{B_z}} - \overset{0(5)}{\cancel{\frac{\partial p}{\partial z}}}\right] + \frac{\mu}{\rho}\left[\underbrace{\cancel{\frac{\partial^2 w}{\partial x^2}}}_{0(6)} + \frac{\partial^2 w}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{0}\right]$$

$$\boxed{\frac{d^2 w}{dy^2} = 0}$$

Y-Momentum:

$$\underbrace{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{0\text{ (continuity)}} = \frac{1}{\rho}\left[B_y - \frac{\partial p}{\partial y}\right] + \frac{\mu}{\rho}\underbrace{\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]}_{0\text{ (continuity)}}$$

$$B_y = \frac{\text{Weight}}{\text{Volume}} = \frac{m \times g}{\text{Volume}} = -\rho g$$

$-$ in $-y$-direction

$$\boxed{\frac{dp}{dy} = -\rho g}$$

X-Momentum:

$$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{0(1)} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{0(6)} + \overbrace{\cancel{v}}^{\text{continuity}}\frac{\partial u}{\partial y} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{0(5)} = \frac{1}{\rho}\left[\overset{0}{\cancel{B_x}} - \overset{0(6)}{\cancel{\frac{\partial p}{\partial x}}}\right] + \frac{\mu}{\rho}\left[\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0(6)} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{0(5)}\right]$$

$$\boxed{\frac{d^2 u}{dy^2} = 0}$$

Energy:

$$\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{0(1)} + \underbrace{u\cancel{\frac{\partial T}{\partial x}}}_{0(6)} + \overbrace{\cancel{v}}^{0\text{(continuity)}}\frac{\partial T}{\partial y} + \underbrace{w\cancel{\frac{\partial T}{\partial z}}}_{0(5)} = \alpha\left[\overset{0(6)}{\cancel{\frac{\partial^2 T}{\partial x^2}}} + \frac{\partial^2 T}{\partial y^2} + \overset{0(5)}{\cancel{\frac{\partial^2 T}{\partial z^2}}}\right] + \overset{0}{\cancel{\frac{\dot{q}}{\rho C_p}}} + \frac{\mu}{\rho C_p}\Phi$$

$$0 = \alpha\frac{d^2 T}{dy^2} + \frac{\mu}{\rho C_p}\Phi$$

Recall:

$$\Phi = 2\left[\cancel{\left(\frac{\partial u}{\partial x}\right)^2}_{0(6)} + \cancel{\left(\frac{\partial v}{\partial y}\right)^2}^{0\ (\text{continuity})} + \cancel{\left(\frac{\partial w}{\partial z}\right)^2}_{0(5)}\right] - \frac{2}{3}\left(\cancel{\vec{\nabla}\cdot\vec{V}}^{0\ (\text{continuity})}\right)^2 + \left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}_{0(6)}\right)^2$$

$$+ \left(\cancel{\frac{\partial v}{\partial z}}_{0(5)} + \cancel{\frac{\partial w}{\partial y}}_{0}\right)^2 + \left(\cancel{\frac{\partial w}{\partial x}}_{0(6)} + \cancel{\frac{\partial u}{\partial z}}_{0(5)}\right)^2$$

(plate not moving in z-direction.)

$$\Phi = \left(\frac{du}{dy}\right)^2$$

Sim

$$\boxed{0 = \frac{d^2T}{dy^2} + \frac{\mu}{k}\left(\frac{du}{dy}\right)^2}$$

Solve ~~press.~~

$$\frac{dp}{dy} = -\rho g \qquad \int_{p_0}^{p} dp = \int_0^y -\rho g\, dy$$

~~Y-mom~~ Y-Momentum

$$p = p_0 - \rho g y$$

Z-Momentum $\quad w = 0!$

X-Momentum

$$\frac{d}{dy}\left(\frac{du}{dy}\right) = 0$$

$$\frac{du}{dy} = C_1 \qquad u = 0 \ @\ y = 0 \Rightarrow C_2 = 0$$

$$u = C_1 y + C_2 \qquad u = U \ @\ y = L \Rightarrow \frac{U}{L} = C_1$$

$$u = U\frac{y}{L}$$

$$\frac{u}{U} = \frac{y}{L}$$

ENERGY EQUATION: $\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2 = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2$

$$d\left(\frac{dT}{dy}\right) = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 dy$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y + C_1$$

$$dT = \left(-\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y + C_1\right) dy$$

$$T = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 \frac{y^2}{2} + C_1 y + C_2$$

$T(y=0) = T_0 \Rightarrow C_2 = T_0$

$T(y=L) = T_1 \Rightarrow T_1 - T_0 = -\frac{\mu}{k}\frac{U^2}{2} + C_1 L$

$$C_1 = \frac{T_1 - T_0 + \frac{\mu}{k}U^2/2}{L}$$

$$T - T_0 = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 \frac{y^2}{2} + \left[\frac{T_1 - T_0 + \frac{\mu}{k}U^2/2}{L}\right] y$$

Let $Ek \equiv$ Ekert # $= \frac{U^2}{C_p(T_1 - T_0)}$ importance of viscous dissipation relative to temp. difference

$Pr = \frac{\mu C_p}{k} = \frac{\mu/\rho}{k/(\rho C_p)} = \frac{\nu}{\alpha}$ — kinematic viscosity — thermal diffusivity

Prandtl #

importance of friction compared to thermal diffusion!

So...

$$T - T_0 = -\frac{\mu}{k}\left(\frac{1}{2}\right)\left[Ek\, C_p(T_1 - T_0)\right]\left(\frac{y}{L}\right)^2 + \left[T_1 - T_0 + \frac{\mu}{k}\left[\frac{Ek\,C_p(T_1 - T_0)}{2}\right]\right]\frac{y}{L}$$

$$\frac{T-T_0}{T_1-T_0} = -\frac{1}{2}\,Ek\,Pr\left(\frac{y}{L}\right)^2 + \frac{y}{L} + \frac{Ek\,Pr}{2}\left(\frac{y}{L}\right)$$

$$= \frac{y}{L} + \frac{Ek\,Pr}{2}\,\frac{y}{L}\left(1-\frac{y}{L}\right)$$

The heat flowing through the wall.—

$$q_{wall} = -k\left.\frac{dT}{dy}\right|_{y=0} = -k\left[\frac{1}{L} + \frac{Ek\,Pr}{2}\left(\frac{1}{L} - 2\frac{y}{L^2}\right)\right]_{y=0} * (T_1-T_0)$$

$$q_{wall} = \frac{-k\,(T_1-T_0)\left(1+\frac{Ek\,Pr}{2}\right)}{L}$$

$$= -\frac{k(T_1-T_0)\left(1+\frac{1}{2}\left(\frac{U^2}{c_p(T_1-T_0)}\right)(Pr)\right)}{L}$$

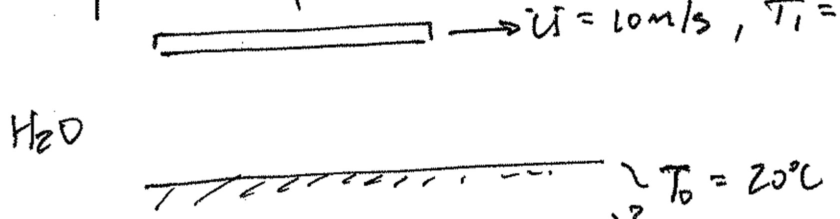
$$= -\frac{k(\cancel{T_1-T_0})\left[\frac{T_1-T_0 + U^2 Pr/(2c_p)}{\cancel{T_1-T_0}}\right]}{L}$$

$$= \frac{k\left[T_0 - \left(T_1 + Pr\,U^2/2c_p\right)\right]}{L}$$

So it's <u>not</u> $q = \frac{k(T_0-T_1)}{L}$ why??

Heat generated by friction! dissipation!

Lets plot temp. dist. for following situation



$$c_p = \frac{4216 + 4183}{2} \doteq 4200 \frac{J}{kg\cdot K}$$

$Pr_{\sim 68°C} = 2.68$

$$Ek = \frac{u^2}{c_p(T_1 - T_0)} = \frac{(10\,m/s)^2}{4200\frac{J}{kg\cdot K}(80\,K)}$$

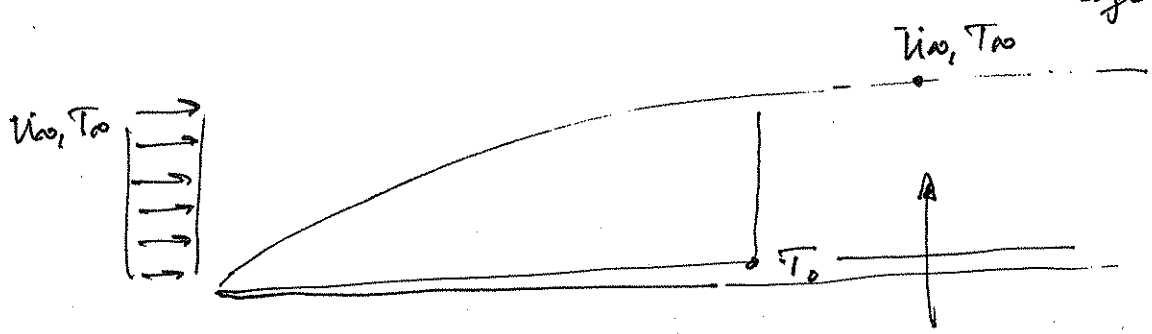
$$= \cancel{2.} \; 3.0 \times 10^{-4}$$

$$\boxed{T^* = y^* + \frac{Ek\,Pr}{2} y^*(1-y^*)}$$

See plot on next page

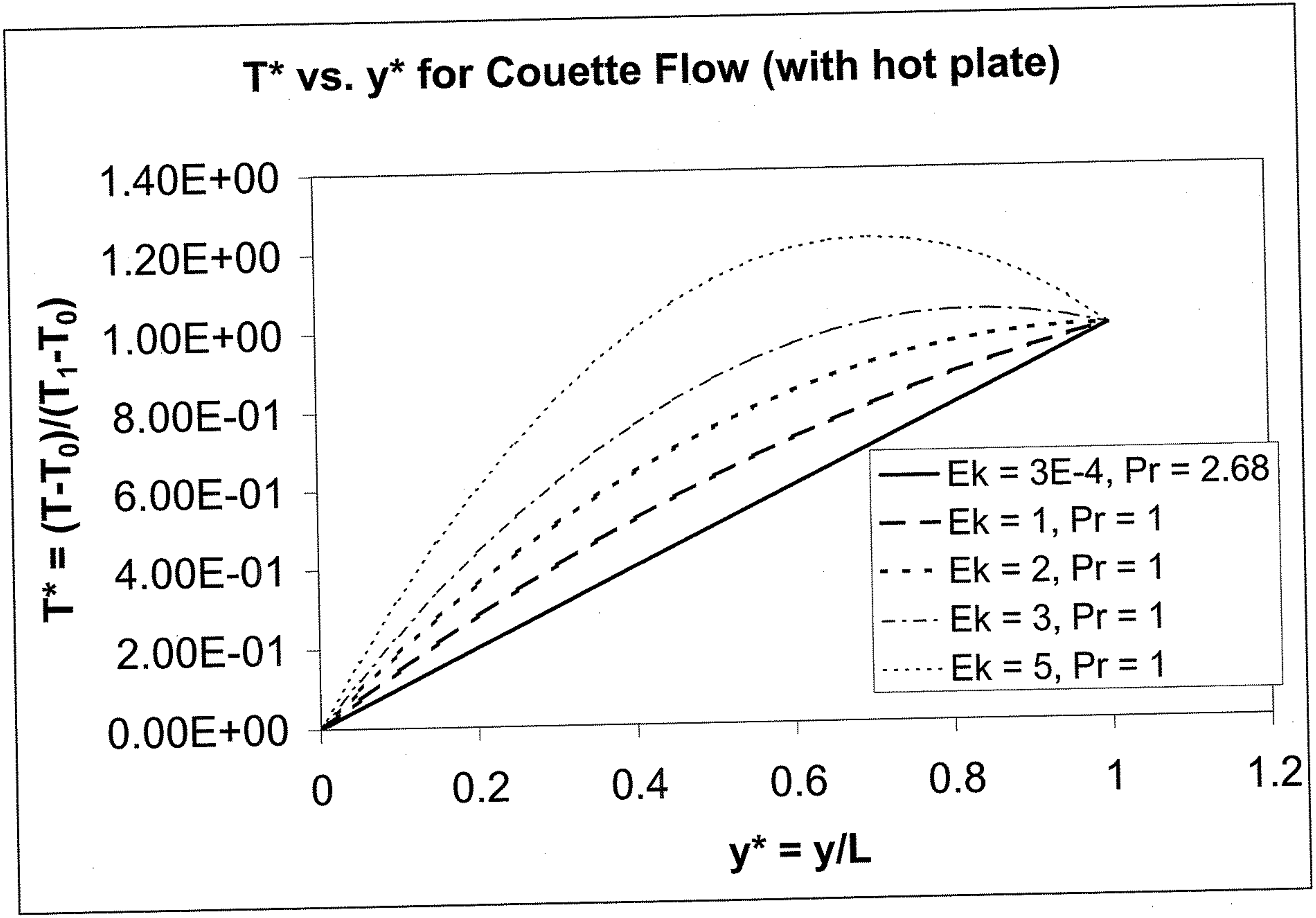
Note, when $Ek \cdot Pr > 2$, temp. between plate & wall may actually be greater than the plate temp. (i.e. heat may actually be flowing into the hot plate!!).

Note, the temp. profiles apply to a boundary layer ... (far from leading edge)



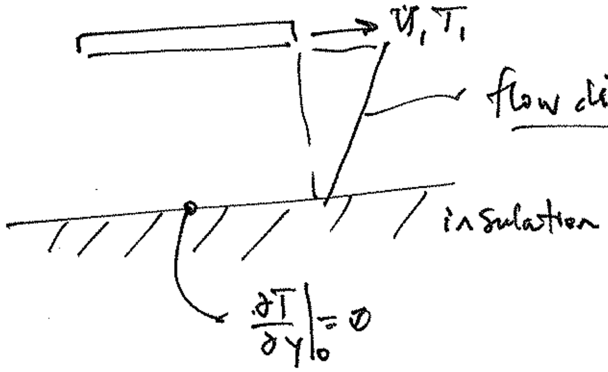
this region is like Couette Flow

6.24(b)



T* vs. y* for Couette Flow (with hot plate)
1.40E+00
1.20E+00
1.00E+00
8.00E-01
6.00E-01
4.00E-01
2.00E-01
0.00E+00
T* = (T-T0)/(T1-T0)
0
0.2
0.4
0.6
0.8
1
1.2
y* = y/L
Ek = 3E-4, Pr = 2.68
Ek = 1, Pr = 1
Ek = 2, Pr = 1
Ek = 3, Pr = 1
Ek = 5, Pr = 1

Let's consider an insulated lower surface



Solution before to Energy Eq. was...

$$T = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 \frac{y^2}{2} + C_1 y + C_2 \qquad T(y=L) = T_1$$

$$\left.\frac{dT}{dy}\right|_0 = 0$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 y + C_1$$

$$\left.\frac{dT}{dy}\right|_0 = 0 \Rightarrow C_1 = 0$$

$$T_1 = -\frac{\mu}{k}\left(\frac{U}{\cancel{L}}\right)^2 \frac{\cancel{L^2}}{2} + C_2$$

$$C_2 = T_1 + \frac{\mu}{k}\frac{U^2}{2}$$

$$T = -\frac{\mu}{k}\left(\frac{U}{L}\right)^2 \frac{y^2}{2} + T_1 + \frac{\mu}{k}\frac{U^2}{2}$$

$$T - T_1 = \frac{\mu U^2}{2k}\left[\left(\frac{y}{L}\right)^2 + 1\right] * \frac{1/\rho c_p}{1/\rho c_p} = \frac{Pr\, U^2}{2c_p}\left[1 - \left(\frac{y}{L}\right)^2\right]$$

adiabatic wall

So... the temp at $y = 0$, $\quad T(0) = T_{aw} = \frac{Pr\, U^2}{2c_p} + T_1$

$q_w = -k \left.\frac{dT}{dx}\right|_0 = 0$!!

Taw (from present example

But remember:
(for $T(0)=T_0$)

$$q_w = \frac{k\left[T_0 - \left(T_1 + \frac{Pr\,U^2}{2c_p}\right)\right]}{L}$$

this is sort of like the temp. diff. in wall between heat flowing w/ To or heat not flowing w/ Tw

$$q_w = \frac{k\left[T_0 - T_{aw}\right]}{L}$$

This is really the driving force for the heat transfer from the wall

~~W~~What about h?

Remember def. ~~for~~ $q = h(T - T_\infty)$

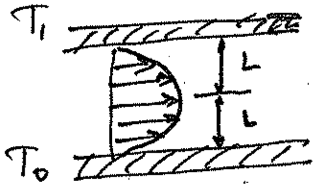
This is meant to be driving temp. difference...

$q_w = h\,(T_0 - T_{aw})$

$$h = \frac{q_w}{T_0 - T_{aw}} = \frac{k}{L}$$

~~Remember... $L = \frac{k}{h}$ distance from wall where all~~

POISEUILLE FLOW



$\frac{dp}{dx} \neq 0$

Assume: ① SS ② laminar flow ③ const. props. ④ const. $\frac{dp}{dx}$ ⑤ $\frac{\partial}{\partial z} = 0$

⑥ $\frac{\partial}{\partial x} = 0$ (excluding p)

Continuity: $\cancel{\frac{\partial u}{\partial x}}^{0} + \frac{\partial v}{\partial y} = 0$ $\quad \frac{dv}{dy} = 0$ $\quad v = \text{const}$ $\quad v = 0$ at the bottom

X-Mom.: $0 = -\frac{1}{\cancel{\rho}}\frac{\partial p}{\partial x} + \frac{\mu}{\cancel{\rho}}\frac{\partial^2 u}{\partial y^2}$ $\quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$

Y-Mom.: $\frac{dp}{dy} = -\rho g$

Z-Mom.: $\frac{d^2w}{dy^2} = 0$

Energy: $\cancel{\frac{\partial T}{\partial t}}$ $\quad 0 = \alpha \frac{d^2T}{dy^2} + \frac{\mu}{\rho c_p}\Phi$ $\quad$ $k/\rho c_p$

$\Phi = \left(\frac{du}{dy}\right)^2$ $\quad$ $\frac{d^2T}{dy^2} + \frac{\mu}{k}\left(\frac{du}{dy}\right)^2 = 0$

Boundary Conditions:

At $y = -L$; $\quad u = 0 = w$, $\quad v = 0$, $\quad T = T_0$

At $y = L$; $\quad u = 0 = w$, $\quad v = 0$, $\quad T = T_1$

X-Motion:

$$\int d\left(\frac{du}{dy}\right) = \int \frac{1}{\mu}\frac{dp}{dx}\,dy$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

$$\cancel{0 = 0 + 0 + C_2}\ , \quad \cancel{0 = \frac{1}{2\mu}\frac{dp}{dx}L}$$

At $y = -L$:

$$0 = \frac{1}{2\mu}\frac{dp}{dx}L^2 - C_1 L + C_2$$

$$C_2 = C_1 L - \frac{1}{2\mu}\frac{dp}{dx}L^2$$

At $y = +L$:

$$0 = \cancel{\frac{1}{2\mu}\frac{dp}{dx}L^2} + C_1 L + C_1 L \cancel{- \frac{1}{2\mu}\frac{dp}{dx}L^2}$$

$$C_1 = 0$$

parabolic profile

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 - \frac{1}{2\mu}\frac{dp}{dx}L^2 = \frac{L^2}{2\mu}\left(-\frac{dp}{dx}\right)\left[1 - \left(\frac{y}{L}\right)^2\right]$$

$$u_{max} = u(0) = \frac{L^2}{2\mu}\left(-\frac{dp}{dx}\right) \qquad \frac{u}{u_{max}} = 1 - \left(\frac{y}{L}\right)^2$$

~~Energy Eq.~~ Also:

$$u_{AVE} = \frac{\int_{-L}^{+L} u\,dy}{\int_{-L}^{+L} dy} = \frac{\frac{L^2}{2\mu}\left(-\frac{dp}{dx}\right)\left[y - \frac{1}{3}\frac{y^3}{L^2}\right]_{-L}^{+L}}{2L}$$

$$= \frac{L}{4\cancel{2}\mu}\left(-\frac{dp}{dx}\right)\left[\left(L - \frac{1}{3}L\right) - \left(-L - \frac{1}{3}(-L)\right)\right]$$

$$= \frac{L}{4\mu}\left(-\frac{dp}{dx}\right)\left[2L - \frac{2}{3}L\right] = \frac{L^2}{3\mu}\left(-\frac{dp}{dx}\right)$$

$$u = \frac{3}{2}u_{AVE}\left[1 - \left(\frac{y}{L}\right)^2\right]$$

6.29

$$\frac{du}{dy} = \frac{L^2}{2\mu}\left(-\frac{dp}{dx}\right)\left[-\frac{2y}{L^2}\right]$$

Energy Eq:
$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left[\frac{L^2}{\cancel{2}\mu}\left(-\frac{dp}{dx}\right)\left(-\frac{\cancel{2}y}{\cancel{L^2}}\right)\right]^2$$

$$= -\frac{\cancel{\mu}}{k}\left(-\frac{dp}{dx}\right)^2\frac{1}{\mu^{\cancel{2}}}\,y^2 \times \frac{L^4}{4\mu} \times \frac{4\mu}{L^4}$$

$$= \left(-\frac{4\mu}{kL^4}\right)\underbrace{\left[\left(-\frac{dp}{dx}\right)\frac{L^2}{2\mu}\right]^2}_{u_m \text{ — max vel. in tube...}} y^2$$

$$= \frac{-4\mu}{kL^4}\,u_m^2\,y^2$$

$$\int d\left(\frac{dT}{dy}\right) = \int \frac{-4\mu}{kL^4}\,u_m^2\,y^2\,dy$$

$$\frac{dT}{dy} = \frac{-4\mu}{kL^4}\,u_m^2\,\frac{1}{3}y^3 + C_1$$

$$T = \frac{-\cancel{4}\mu}{3kL^4}\,u_m^2\,\frac{1}{\cancel{4}}y^4 + C_1 y + C_2$$

B.C.'s:
$$T_0 = \frac{-\mu}{3k\cancel{L^4}}\,u_m^2\,\cancel{L^4} - C_1 L + C_2$$

$$C_2 = T_0 + C_1 L + \frac{\mu}{3k}\,u_m^2$$

$$T_1 = \frac{-\mu}{3k\cancel{L^4}}\,u_m^2\,\cancel{L^4} + C_1 L + T_0 + C_1 L + \frac{\mu}{3k}\,u_m^2$$

$$C_1 = \frac{T_1 - T_0}{2L}$$

$$T = \frac{-\mu}{3kL^4}\,u_m^2\,y^4 + \frac{T_1 - T_0}{2L}\,y + T_0 + \frac{T_1 - T_0}{2\cancel{L}}(\cancel{L}) + \frac{\mu}{3k}\,u_m^2$$

$$T = T_0 + \frac{T_1 - T_0}{2}\left(1 + \frac{y}{L}\right) + \frac{\mu}{3k} u_m^2 \left[1 - \left(\frac{y}{L}\right)^4\right]$$

$$T_* = \frac{T - T_0}{T_1 - T_0} = \frac{1}{2}(1 + y_*) + \frac{1}{3}\overbrace{\frac{\mu/\rho}{k/\rho c_p}}^{Pr}\ \overbrace{\frac{u_m^2}{c_p(T_1 - T_0)}}^{Ek}\left[1 - \left(\frac{y}{L}\right)^4\right]$$

$$\boxed{T_* = \frac{1}{2}(1 + y_*) + \frac{Ek \cdot Pr}{3}\left[1 - y_*^4\right]}$$

See plot next page... Note $Ek \cdot Pr > \frac{3}{8}$ ...

you get heat flow <u>into</u> the upper plate!

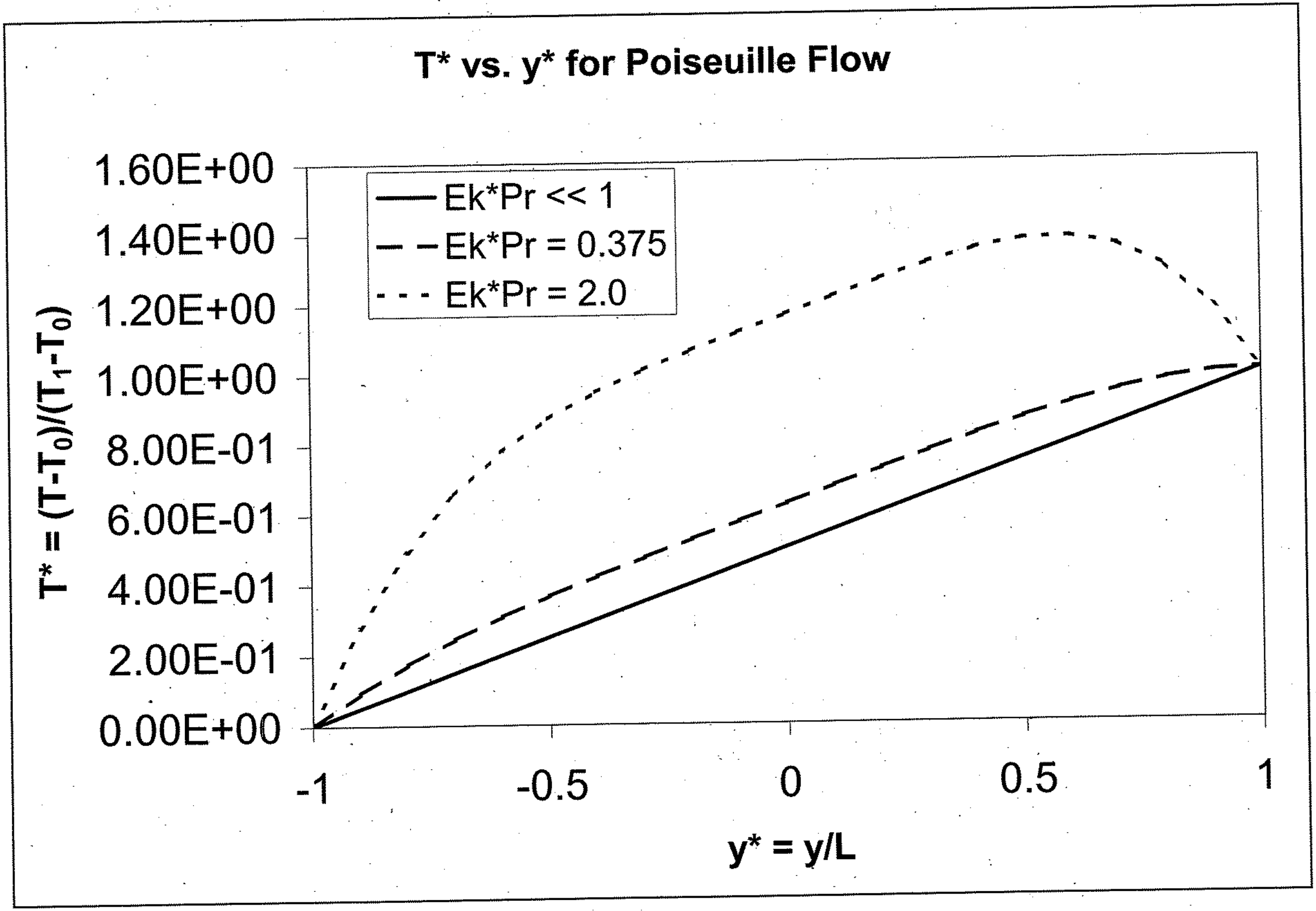
$$q_w = \frac{k}{2L}\left(T_0 - T_1 - \frac{8}{3}\frac{Pr\, u_m^2}{c_p}\right)$$

which can be rewritten as

adiabatic wall temp.

$$q_w = \frac{k}{2L}\left(T_{wall} - T_{aw}\right)$$

6.30(b)


T* vs. y* for Poiseuille Flow
Ek*Pr << 1
Ek*Pr = 0.375
Ek*Pr = 2.0
1.60E+00
1.40E+00
1.20E+00
1.00E+00
8.00E-01
6.00E-01
4.00E-01
2.00E-01
0.00E+00
T* = (T-T0)/(T1-T0)
-1
-0.5
0
0.5
1
y* = y/L